

Finite time singularity of the stochastic harmonic map flow

Antoine Hocquet*

CMAP, Ecole Polytechnique, CNRS,
Université Paris Saclay, 91128, Palaiseau, FRANCE

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Abstract

We investigate the influence of an infinite dimensional Gaussian noise on the bubbling phenomenon for the stochastic harmonic map flow $u(t, \cdot) : \mathbb{D}^2 \rightarrow \mathbb{S}^2$, from the two-dimensional unit disc onto the sphere. The diffusion term is assumed to have range one pointwisely in the tangent space $T_{u(t,x)}\mathbb{S}^2$, so that the noise preserves the 1-corotational symmetry of solutions. Under the assumption that its space-correlation is of trace class (in some appropriate Hilbert space), we prove that the noise generates blow-up with positive probability. This scenario happens no matter how we choose the initial data, provided it fulfills the latter symmetry assumption.

1 Introduction and main result

1.1 Motivations

Effect of a noise term on the appearance of a finite time singularity has already been investigated for several stochastic PDE's, including the Schrödinger equation [18, 20] where it is shown to generate blow-up with positive probability, for any initial data. Some results in the same spirit have been obtained for the stochastic heat equation [34, 33, 22], and also for the so-called Dyadic Model [40], where the author shows in addition the ineluctability of the blow-up. Our work comes from an attempt to understand the effect of noise on the bubbling phenomenon for the two-dimensional Stochastic Landau-Lifshitz-Gilbert equation, for which the Stochastic Harmonic map flow is a (drastically) simplified model. The corresponding deterministic equation (LLG) has first arisen in [31] as a purely phenomenological model for magnetization dynamics, and later Gilbert [26] proposed a Lagrangian formulation of the work of L. Landau and E. Lifshitz. Meanwhile, W.F Brown had created in the 50's the theory of micromagnetism (see the celebrated monograph [10]), where the magnetization

* antoine.hocquet@cmap.polytechnique.fr

of a ferromagnetic material $\mathcal{M} \subset \mathbb{R}^3$ is represented as a time-dependent continuum $u : [0, T] \times \mathcal{M} \rightarrow \mathbb{S}^2$. Having set all physical constants to 1, the vector field u is, at equilibrium, a solution of a minimization problem for the so-called Brown energy, under the pointwise constraint $|u(t, x)| = 1$, a.e. In this framework, the deterministic harmonic map flow from \mathcal{M} to the unit sphere \mathbb{S}^2 , namely

$$\begin{cases} \partial_t u = \Delta u + u|\nabla u|^2, & \text{on } [0, T] \times \mathcal{M}, \\ u = \varphi, & \text{on } [0, T] \times \partial\mathcal{M} \cup \{0\} \times \mathcal{M}, \end{cases} \quad (\text{HMF})$$

can be obtained by assuming that:

- the Brown energy equals the exchange energy $E \equiv (1/2) \int_{\mathcal{M}} |\nabla u|^2 d\mathcal{M}$ (corresponding to closest neighbour interaction);
- the system is “overdamped”: there is no precession of u around the effective field $H_{\text{eff}}(u) := -\nabla E(u) \equiv \Delta u$.

Note that (HMF) is in fact the gradient flow associated to E , as the right hand side equals the pointwise orthogonal projection of the effective field onto $\text{Vect } u(t, x)^\perp$. This model has been independently studied e.g. in [25, 29, 23, 24], in the more general case where $u : \mathbb{R}_+ \times \mathcal{M} \rightarrow \mathcal{N}$ and \mathcal{M}, \mathcal{N} are Riemannian manifolds. The interest of this evolution problem lies in the fact that it provides a tool to construct harmonic maps, in some given homotopy class. Recall that a map $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ is said to be *harmonic* if it is a regular solution to the minimization problem associated to the energy $(1/2) \int_{\mathcal{M}} |\nabla u|_{T_u \mathcal{N}}^2$.

Blowing-up of symmetric solutions in 2D. The case where \mathcal{M} denotes a surface has been particularly investigated in the literature, since the latter H^1 energy barely fails to control the nonlinear terms in the equation, and thereby to globalize the local solutions. If the initial energy is less than some quantum

$$E(\varphi) < \epsilon_1, \quad (1.1)$$

depending on \mathcal{M} only, then K.C. Chang [30] has shown, generalizing the ideas of J. Eells, J.H. Sampson [25] and M. Struwe [41], that the solution $u(t, \cdot)$ of (HMF) is global and uniformly converges towards an harmonic map u_∞ as $t \rightarrow \infty$, homotopic to φ .

Oppositely, the local solution u of (HMF) may not be defined globally (in the classical sense) if (1.1) is not fulfilled. Examples of finite-time blowing-up solutions were first obtained by J.-M. Coron and J.-M. Ghidaglia in [16], in the case $\mathcal{M} = \mathbb{R}^n$, and $N = \mathbb{S}^n$ with $n \geq 3$. Later, K.C. Chang, W. Ding and R. Ye [15] gave explicit blowing-up solutions for the case $u : [0, T] \times \mathbb{D}^2 \equiv \{x \in \mathbb{R}^2 : |x| < 1\} \rightarrow \mathbb{S}^2$, considering 1-corotational solutions¹ of the form $u = u_h$ with

$$u_h(t, x) := \left(\frac{x}{|x|} \sin h(t, |x|); \cos h(t, |x|) \right). \quad (1.2)$$

¹In the existing literature, these maps are often called “equivariant”, or 1-equivariant, although the latter can have by definition an additional degree of freedom b , so that $u(r \cos \theta, r \sin \theta) = R_\theta (a(r), b(r), c(r))$ with $a^2 + b^2 + c^2 \equiv 1$, R_θ corresponding to the rotation of angle θ and axis \vec{k} . The form given above corresponds to the special case where $a(r) = \sin h(r)$, $b(r) = 0$, $c(r) = \cos h(r)$ and should be rather called “1-corotational” (see for instance [7]).

Under the latter symmetry assumption, the system (HMF) can then be reduced to a parabolic equation on the scalar map $h(t, r)$:

$$\begin{cases} \partial_t h = \partial_{rr} h + \frac{\partial_r h}{r} - \frac{\sin 2h}{2r^2} , & \text{for } (t, r) \in [0, T] \times (0, 1) , \\ h(t, 0) = 0 , \quad h(t, 1) = \gamma , & \text{for } t \in [0, T] , \\ h(0, r) = h_0(r) , & \text{for } r \in (0, 1) , \end{cases} \quad (1.3)$$

and a comparison principle for (1.3) can be stated. In [15], the authors exhibit a class of self-similar, blowing-up subsolutions of the parabolic problem (1.3), implying the divergence of $\partial_r h(t, 0)$, at some finite time $t_* > 0$. As described by the results of M. Struwe [41], this implies blow-up for the corresponding solution u of (HMF), with “forward bubbling”. Roughly speaking: as $t \nearrow t_*$, an amount of energy ϵ_1 localizes at the center of the disc – note that the assumption (1.2) prevents the bubbling to take place elsewhere than at the center, otherwise we would have a singular annulus with infinitely energy on it.

In (1.3), the number γ is the angle between $\varphi|_{|x|=1}$ and vertical axis, so that in the case where

$$\varphi|_{\partial\mathbb{D}^2} = \vec{k} := (0, 0, 1) , \quad (1.4)$$

we have $\gamma = 0$. K.C. Chang-W. Ding-R. Ye’s result has been stated under the assumption $h(t, 1) \equiv \gamma > \pi$. Nevertheless, a similar conclusion was established in [9] for homogeneous Dirichlet boundary conditions, which is the context of our study. Since they are not given explicitly in that form, we summarize below the existing results.

Theorem: finite-time singularity for HMF ([15, 9]). *Assume $\gamma = 0$. There exists $\bar{h}_0 \in C^1([0, 1])$ with $\bar{h}_0(0) = \bar{h}_0(1) = 0$, such that every solution h of (1.3) with $h_0 \geq \bar{h}_0$ a.e. , leaves C^1 in finite time: for some $t_*(h_0) > 0$, the associated 1-corotational solution u_h verifies $\lim_{t \rightarrow t_*} |\nabla u_h(t, 0)| = \infty$.*

Stability under random perturbations. The stability of the above result under small perturbations of the initial data, although being a seemingly academic question, echoes issues related to the appearance of singularities in the 2D stochastic LLG, for which blow-up is numerically observed in [6]. Uniqueness of solutions is also questioned through this topic, since in [9] the authors have constructed examples of nonuniqueness for the weak solutions of (HMF), by using different ways of extending the solution after the singular time.

Non-equilibrium micromagnetism acquires new levels of complexity when taking into account the effects due to temperature. Thermal effects were first considered in [35], and then formalized through Gaussian white noise in time in [11], when restricting to a monodomain particle (or equivalently if we assume constant magnetization in space: $u(t, x) = U(t)$). In this context, the derivative $\frac{dB}{dt}$ of a 3D Brownian Motion has to be added to the effective field, see [11]. This remains true for \mathcal{M} composed of n finitely adjacent domains (on which magnetization is constant), for which the corresponding vector $(U^i)_{i \leq n}$ formally satisfies:

$$dU^i = -U^i \times \left(U^i \times \left(H_{\text{eff}}^i + \frac{dB_i}{dt} \right) \right) + U^i \times \left(H_{\text{eff}}^i + \frac{dB_i}{dt} \right), \quad 1 \leq i \leq n$$

(see [8]), where B^1, \dots, B^n are uncorrelated. In the case of a continuous spin chain $u = u(t, x)$, the latter observations lead to Gaussian space-time white noise $\xi = \frac{dW}{dt}$, $t \in \mathbb{R}_+ \mapsto W(t) \in L^2(\mathcal{M})$ being a cylindrical Wiener process. Using that $u \times (u \times \Delta u) = -\Delta u - u|\nabla u|^2$, and dropping the noise in the first term for simplicity (see however [36] for a justification that it leads to the same pointwise statistics), the stochastic Landau-Lifshitz-Gilbert equation writes under the form:

$$\partial_t u = \Delta u + u \times \Delta u + u|\nabla u|^2 + u \times \frac{dW}{dt}, \text{ on } [0, T] \times \mathcal{M}, \quad (\text{SLLG})$$

where $u \times$ denotes the vector product, and where various conditions on the parabolic boundary can be considered.

There has been recently a series of papers on (SLLG) in two or three-dimensional domains [13, 12, 5, 4, 27, 2] dealing essentially with the notion of “weak martingale solution”, that is weak both in the probabilistic sense, and in the PDE sense: solutions are constructed, considering the Wiener process as an unknown of the problem, and they belong pathwisely to the space $C([0, T]; L^2) \cap L^\infty([0, T]; H^1)$. However, further questions related to uniqueness and partial regularity of solutions seem quite difficult to answer in 3D, since weak solutions are not unique in general for the deterministic equation [3], even under the assumption that the energy decreases along the flow.

Fixing $\mathcal{M} := \mathbb{D}^2$, the two-dimensional unit disc, the model we consider here is the stochastic partial differential equation:

$$\begin{cases} du = (\Delta u + u|\nabla u|^2) dt + u \times \circ dW, & \text{on } [0, T] \times \mathbb{D}^2, \\ u = \varphi, & \text{on } \{0\} \times \mathbb{D}^2 \cup [0, T] \times \partial \mathbb{D}^2, \end{cases} \quad (\text{SHMF})$$

where $W = (w_1, w_2, w_3)$ denotes a Wiener Process in the space $L^2(\mathbb{D}^2; \mathbb{R}^3)$, whereas “ \circ ” means that the Stratonovitch rule is used. It is well-known that parabolic equations of the form (SHMF) can be ill-posed in dimension two, see e.g. [28], and therefore the case of space-time white noise will not be treated in this article. Besides, we need enough regularity in space so that blow-up actually makes sense. We will build solutions that are strong in the probabilistic sense (mild solutions), and sufficiently regular in space, so that the singular time τ corresponds to the first moment when the solution leaves $C^1(\mathbb{D}^2)$. Further assumptions on the spatial correlation $\mathbb{E}[W(t, x) \cdot W(t', x')]$ will be done below.

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1.2 Main result and comments

Writing the unknown as

$$u = (\cos g \sin h, \sin g \sin h, \cos h), \quad (1.5)$$

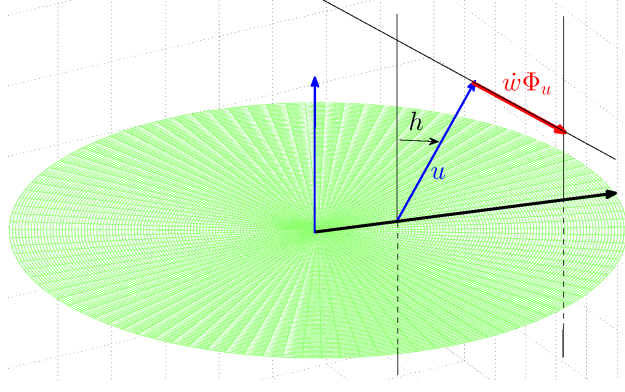


Figure 1: “1-corotational noise”, represented in red.

where $g, h : [0, T] \times \mathbb{D}^2 \rightarrow \mathbb{R}$ and denoting by (Φ_u, Θ_u) the co-rotating frame associated to its coordinates (g, h) , namely

$$\begin{cases} \Phi_u := (\cos g \cos h, \sin g \cos h, -\sin h) , \\ \Theta_u = (-\sin g, \cos g, 0) , \end{cases} \quad (1.6)$$

then it can be formally checked that (SHMF) and the equation

$$du = (\Delta u + u|\nabla u|^2) dt + dw_1 \circ \frac{\Theta_u}{\sin h} + dw_2 \circ \Phi_u \quad (1.7)$$

lead to the same Kolmogorov equation on the law of the solution (for a proof in finite dimension see [39]). There is no hope however to preserve 1-corotational symmetry along the flow if a part of du acts in the direction $\text{Vect} \Theta_u = \{(x_1, x_2, 0), \vec{k}\}^\perp$, so that we replace w_1 by 0, denote by $w := w_2$ which is *scalar* (but still infinite dimensional), and also assume that $w(t, x)$ *depends only on the radius* $r = |x|$, see Figure 1.

We will consider noise that is regularized in space, namely with a covariance operator $\phi\phi^*$ of trace class in the separable Hilbert space

$$H := \left\{ f : [0, 1] \rightarrow \mathbb{R} : |f|_H^2 := \int_0^1 f(r)^2 r dr < \infty \right\}. \quad (1.8)$$

The equation we consider in this article writes

$$\begin{cases} du = (\Delta u + u|\nabla u|^2) dt + \Phi_u \circ dw_\phi , \\ u(0) = u_{h_0} , \end{cases} \quad (\text{SHMF}') \quad (1.9)$$

for a Wiener process $w_\phi(t, |x|) := \sum B_k(t) \phi e_k(|x|)$, where (e_k) denotes an ONB of H , and the B_k 's are real-valued Brownian motions. Note that, using the notation (1.2), we have $\Phi_u = u_{h+\pi/2} = (\frac{x}{|x|} \cos h(t, |x|); -\sin h(t, |x|))$.

Denoting by $(-\Delta)^\alpha$, $\alpha \in \mathbb{R}$, the fractional powers of the Laplace operator associated to Dirichlet boundary conditions, and letting \dot{H}^α be the completion of $C_0^\infty(\mathbb{D}^2)$ for the norm $|u|_{\dot{H}^\alpha} := (\int_{\mathbb{D}^2} |(-\Delta)^{\alpha/2} u(x)|^2 dx)^{1/2}$, our main result states as follows.

Theorem 1. *Let w_ϕ be a H -valued Wiener process, and denote by $\phi : H \rightarrow H$ the square-root of its covariance operator.*

Assume that for some $\beta \in (2, 4)$:

- $\sum_{k \in \mathbb{N}} |\phi e_k(|x|)|_{\dot{H}^\beta}^2 < \infty$ for every orthonormal basis $(e_k)_{k \geq 0}$ of H ;
- $\ker \phi^* = \{0\}$;
- with the notation (1.2), we have $u_0 = u_{h_0}$ for some $h_0 \in H$, and $|u_0|_{\dot{H}^\beta} < \infty$.

Then, there exists a unique maximal local solution u of (SHMF') with continuous trajectories in \dot{H}^β , a.s. Moreover denoting by τ^β its maximal time of existence, then for every $t_* > 0$:

$$\mathbb{P}(\tau^\beta \leq t_*) > 0.$$

Additionally, for every $\beta_* > 2$, there holds:

$$\tau^\beta \leq t_* \implies \limsup_{t \rightarrow \tau^\beta} |u(t)|_{\dot{H}^{\beta_*}} = \infty.$$

Remark 1.1. More information on blow-up behaviour for the deterministic equation (HMF) has been obtained during the last years, concerning the stability/instability of such dynamics – see the series of works [7, 32, 38], based on the formal asymptotics in [42]. In [32], the authors show the existence, but instability, of initial data leading to blow-up for the Heisenberg equation $\partial_t u = u \times \Delta u$, under the assumption that u is 1-equivariant. It is proved that instability is due to the extra degree of freedom compared to the 1-corotational case. This additional degree is necessary, for the Heisenberg equation as well as for the full deterministic Landau-Lifshitz-Gilbert problem, namely $\partial_t u = \Delta u + u|\nabla u|^2 + u \times \Delta u$ (LLG). It corresponds to allow for a general map $g = g(t, r)$ in (1.5). The numerical experiments in [43] also evidence the fact that for (LLG), the “pre-blow-up set” (namely the initial data leading to blow-up) forms a codimension one set only.

Oppositely, the overdamped model (HMF) allows to reduce the equation to a scalar problem (1.3), i.e. with only one degree of freedom. In this context, Raphaël and Schweyer (in case where \mathbb{D}^2 is replaced by \mathbb{R}^2) have shown that this set is stable under small perturbations *in the direction preserving 1-corotational symmetry*. Indeed, it is shown in [38] that if $\mathcal{Q} = \mathcal{Q}(t, x)$ denotes the least energy harmonic map, namely $\mathcal{Q}(t, x) = \left(\sin Q(|x|) \frac{x}{|x|}, \cos Q(|x|) \right)$, where for $r \geq 0$, $Q(r) = 2 \arctan(r)$, then there exists an open set \mathcal{O} of 1-corotational initial data of the form

$$v_0 = \mathcal{Q} + \varepsilon_0, \quad \varepsilon_0 \in \mathcal{O} \subset \dot{H}^1 \cap \dot{H}^4,$$

such that the corresponding solutions v to (HMF) blow up in finite time $T(v_0)$. Theorem 1 is coherent with the stability result above, for the noise term in (SHMF') does not affect the 1-corotational symmetry.

Outline of the proof. Denoting by Σ the parabolic boundary $\Sigma := \{0\} \times [0, 1] \cup [0, T] \times \{0, 1\}$, then a formal application of the Itô formula shows that (SHMF') writes as an equation on the colatitude h of u :

$$dh = \left(\partial_{rr} h + \frac{\partial_r h}{r} - \frac{h}{r^2} + \frac{2h - \sin 2h}{2r^2} \right) dt + dw_\phi, \quad \text{on } [0, T] \times [0, 1], \quad (1.9)$$

where $(h - h_0)|_\Sigma = 0$. Due to compensations, when h is solution, we may have $\int_0^1 (\partial_r h / r - \sin 2h / (2r^2))^2 r dr < \infty$ even if both terms of the integrand are not summable separately. This integral behaves as $\int_0^1 (\partial_r h / r - h / r^2)^2 r dr$, which is the reason why we write

the linear part of the equation as $A := \partial_{rr} + (\partial_r/r - 1/r^2)$. Note that the noise is *additive*, and thus we have $h = v + z$, where v solves the *perturbed equation*:

$$\partial_t v = \partial_{rr} v + \frac{\partial_r v}{r} - \frac{v}{r^2} + \frac{2z + 2v - \sin 2(v + z)}{2r^2}, \quad \text{on } [0, T] \times [0, 1], \quad (1.10)$$

with $(v - h_0)|_\Sigma = 0$, and where $z = z(t, r)$ denotes a generic trajectory $Z(\omega)$ in the support of the solution of the stochastic linear equation

$$dZ = \left(\partial_{rr} Z + \frac{\partial_r Z}{r} - \frac{Z}{r^2} \right) dt + dw_\phi, \quad Z|_\Sigma = 0.$$

Denote by $h = h(h_0, Z)$ the local solution $v + Z$ of (1.9). Theorem 1 is a consequence of the existence of a “nice” pre-blow-up set \mathfrak{H} , namely a set of initial data h_0 such that: (a) states in \mathfrak{H} are reachable by the Markov Chain $h(h_0, Z, t)$ (in a sense precised below); (b) the solutions starting from $h_0 \in \mathfrak{H}$ blow up in finite time, with positive probability.

The local solvability is obtained in Sec. 2, where we prove the property (a). Theorem 1 is then obtained as a consequence of (b) (whose precise statement is Lemma 2). Lemma 2 is the core of the argument; its proof will be done in sec. 3. Technical facts related to local solvability and the comparison principle for (1.10) are treated in the appendix.

1.3 Notation and framework

In the sequel we denote by I the compact interval $[0, 1]$. For $1 \leq p < \infty$, the notation L^p_{rdr} will be used to designate the Banach space of real valued measurable maps $r \mapsto f(r)$, $r \in I$, such that $\|f\|_{L^p_{rdr}} := (\int_0^1 |f(r)|^p r dr)^{1/p} < \infty$. The special case $H = L^2_{rdr}$, $|\cdot|_H$, defines a Hilbert space for the inner product $f, g \in H \mapsto \langle f, g \rangle = \int_0^1 f(r)g(r)r dr$.

We need to introduce some functional spaces. Let A be the self-adjoint operator on H given by

$$D(A) = \left\{ f \in H : \int_0^1 (\partial_{rr} f)^2 + \left(\frac{\partial_r f}{r} - \frac{f}{r^2} \right)^2 r dr < \infty \right\}, \quad (1.11)$$

$$Ah = \partial_{rr} h + \left(\frac{1}{r} \partial_r - \frac{1}{r^2} \right) h, \quad f \in D(A). \quad (1.12)$$

This operator has eigenpairs $\{(e_k, \lambda_k), k \geq 1\}$ with (e_k) forming an orthonormal basis of H , while the values λ_k are negative and asymptotically quadratic in k – see Appendix A.1. We can define, when $\beta \in \mathbb{R}$, the fractional power $(-A)^{\beta/2}$ through

$$(-A)^{\beta/2} h := \sum_{k \in \mathbb{N}} (-\lambda_k)^{\beta/2} \langle h, e_k \rangle e_k, \quad (1.13)$$

for every $h \in V_\beta$ where

$$\left\{ h \in H, \quad |h|_\beta^2 := \sum_{k \in \mathbb{N}} (-\lambda_k)^\beta \langle h, e_k \rangle^2 < \infty \right\}. \quad (1.14)$$

For $\beta \in \mathbb{R}$, the norm in $C([0, T]; V_\beta)$ (i.e. the space of continuous functions with values in V_β), will be denoted by the double bars $\|\cdot\|_{T, \beta}$, namely if $z \in C([0, T]; V_\beta)$ we write

$$\|z\|_{T, \beta} := \sup_{0 \leq t \leq T} |z(t)|_\beta. \quad (1.15)$$

In the whole paper, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ satisfying the usual conditions. Note that the couples $(V_\beta, |\cdot|_\beta)$ form separable Hilbert spaces, and thus by the classical theory of SPDE's [17], the adapted H -valued Wiener process

$$w_\phi(t) = \sum_{k \in \mathbb{N}} B_k(t) \phi e_k, \quad (1.16)$$

where $(B_k)_{k \in \mathbb{N}}$ stands for a sequence of real-valued independent brownian motions in time, $(e_k)_{k \in \mathbb{N}}$ is an ONB of H , and $\phi : H \rightarrow V_\beta$ is a Hilbert-Schmidt operator, has continuous paths in the space V_β , with full probability.

The space of Hilbert-Schmidt operators from H into some Hilbert space K will be denoted by $\mathbb{L}_2(H; K)$.

Remark 1.2. For $f \in H$, if $x \equiv r(\cos \theta, \sin \theta) \in \mathbb{D}^2$, define $F : \mathbb{D}^2 \rightarrow \mathbb{R}^2$ by $F(x) = (f(r) \cos \theta, f(r) \sin \theta)$. We have $|f|_H = (2\pi)^{1/2} |F|_{L^2(\mathbb{D}^2; \mathbb{R}^2)}$ and if $f \in V_2$, then $F \in D(\Delta) \equiv H^2 \cap H_0^1$ with $\Delta F = (Af(r) \cos \theta, Af(r) \sin \theta)$. Plugging the ansatz above in $\nabla^2 F$, there holds in addition:

$$\int_{\mathbb{D}^2} |\nabla^2 F|^2 dx = 2\pi \int_0^1 (\partial_{rr} f)^2 r dr + 4\pi \int_0^1 \left(\frac{\partial_r f}{r} - \frac{f}{r^2} \right)^2 r dr. \quad (1.17)$$

By a classical inequality, (1.17) justifies that the norms $|\partial_{rr} f|_H + \left| \left(\frac{\partial_r}{r} - \frac{1}{r^2} \right) f \right|_H$ and $|f|_2 \equiv |Af|_H$, are in fact equivalent on V_2 .

Remark 1.3. For $p \in [1, \infty)$, $\beta \in \mathbb{R}$, $f \in V_\beta$, if $\beta < 1$ and if

$$1 \leq p \leq p^* = \frac{2}{1-\beta},$$

the classical Sobolev Embedding Theorem in dimension 2 (see [1]) implies that $|F|_{L^p(\mathbb{D}^2)} \lesssim |(-\Delta)^{\beta/2} F|_{L^2(\mathbb{D}^2)}$, where we use the notations of Remark 1.2. Since $|F|_{L^p(\mathbb{D}^2)} = (2\pi)^{\frac{1}{p}} |f|_{L_{rdr}^p}$, and $|(-\Delta)^{\beta/2} F|_{L^2(\mathbb{D}^2)} = |f|_\beta$, it is straightforward that we have the continuous embedding: $V_\beta \hookrightarrow L_{rdr}^p$. Similarly if $\beta > 1$, then $V_\beta \hookrightarrow C(I; \mathbb{R})$. In addition, by the formula

$$|\nabla F|^2 = (\partial_r f)^2 + \frac{f^2}{r^2}, \quad (1.18)$$

then for any $\beta > 2$, there exists $c_\beta > 0$ such that for all $f \in V_\beta$, $|\partial_r f|_{L^\infty} \leq c_\beta |f|_\beta$.

2 Proof of Theorem 1

2.1 Local solvability

Given $\beta \geq 0$ and $h_0 \in V_\beta$ equation (1.9) can be written as an infinite dimensional SDE in the space V_β :

$$\begin{cases} dh = (Ah + b(r, h(r))) dt + dw_\phi, & \text{for } t \in \mathbb{R}^+, \\ h(0) = h_0, \end{cases} \quad (2.1)$$

where “d” denotes Itô differential, whereas the term $b(r, h(r))$ denotes the nonlinearity

$$b(r, h(r)) = \frac{2h(r) - \sin 2h(r)}{2r^2}, \quad r \in I \setminus \{0\}. \quad (2.2)$$

This defines an element of H if $\beta \geq 4/3$, as will be shown in the proof of Proposition 1 below.

Proposition 1 (Existence, uniqueness and regularity of the solutions). *Let $4 > \beta > 4/3$, and take $\phi \in \mathbb{L}_2(H, V_\beta)$. Then, for $h_0 \in V_\beta$, there exist a stopping time $\tau^\beta(h_0)$, and a unique h with paths in $C([0, \tau^\beta]; V_\beta)$, a.s., mild solution of (2.1) in the sense that*

$$h(t) = S(t)h_0 + \int_0^t S(t-s)b(\cdot, h(s))ds + \int_0^t S(t-s)dw_\phi(s), \text{ for } t \in [0, \tau^\beta], \text{ a.s.}, \quad (2.3)$$

$S(\cdot)$ being the semigroup $e^{\cdot A}$. The stopping time τ^β is maximal, namely the solution cannot be extended continuously in V_β after τ^β .

Moreover, the regularity propagates in the sense that

$$\tau^\beta < \infty \implies \limsup_{t \rightarrow \tau^\beta(h_0)} |h(t)|_{\beta_*} = \infty \text{ for every } \beta_* > 2$$

(in particular $\tau^\beta = \tau^{\beta_*}$).

Remark 2.1. It is classical that a mild solution of (2.1) is also a weak solution, namely we have a.s. for $t < \tau(h_0, \omega)$: $\langle h(t) - h_0, \zeta \rangle = \int_0^t \langle Ah + b(r, h), \zeta \rangle + \int_0^t \langle dw_\phi, \zeta \rangle$, for every $\zeta \in C_0^\infty(I)$ – see e.g. [17, chap. 6]. Using the notation (1.2), the map $h \in \mathbb{R} \mapsto u_h \in \mathbb{R}^3$ has bounded derivatives up to second order, which by Lebesgue Theorem implies the same property for the functional $\Phi(h) := \iint_{I \times [0, 2\pi]} u_{h(r)} \cdot \psi(r \cos \theta, r \sin \theta) r dr d\theta$, where ψ is any element of $C_0^\infty(\mathbb{D}^2; \mathbb{R}^3)$. More precisely we have for $h, k, \ell \in H$:

$$D\Phi(h)k = \iint_{I \times [0, 2\pi]} k(r)u_{h(r)+\pi/2} \cdot \psi(r \cos \theta, r \sin \theta) r dr d\theta, \\ D^2\Phi(h)[k, \ell] = - \iint_{I \times [0, 2\pi]} k(r)\ell(r)u_{h(r)} \cdot \psi(r \cos \theta, r \sin \theta) r dr d\theta.$$

If we assume that the process h has trajectories supported in $C([0, \tau]; V_2)$, then denoting by $u(t, x) := u_h(t, |x|)$ where $x \in \mathbb{D}^2$, by $(f, g)_{L^2} := \int_{\mathbb{D}^2} f(x) \cdot g(x) dx$, writing Itô formula and changing the variables gives:

$$(u(t), \psi)_{L^2} = \int_0^t (u_0, \psi)_{L^2} ds + \int_0^t \left(\left(\partial_{rr} h + \frac{\partial_r h}{|x|} - \frac{h}{|x|^2} \right) u_{h+\pi/2}, \psi \right)_{L^2} ds \\ - \frac{1}{2} \sum_{k \geq 0} \int_0^t (u(\phi e_k)^2, \psi)_{L^2} ds + \int_0^t (u_{h+\pi/2} dw_\phi, \psi)_{L^2}. \quad (2.4)$$

Noticing furthermore that for $u = u_h$ the term $\Delta u + u|\nabla u|^2$ equals $(\partial_{rr} h + \frac{\partial_r h}{r} - \frac{\sin 2h}{r^2}) u_{h+\pi/2}$, together with the Stratonovitch-to-Itô rule:

$$(u_{h+\pi/2} \circ dw_\phi, \psi)_{L^2} = (u_{h+\pi/2} dw_\phi, \psi)_{L^2} - \frac{1}{2} \sum_{k \geq 0} (u(\phi e_k)^2, \psi)_{L^2} ds,$$

we obtain that u is indeed a weak solution of (SHMF').

The case where h is supported in $C([0, \tau]; V_\beta)$ for $4/3 < \beta < 2$ would lead to a similar conclusion, e.g. by adapting the generalized Itô formula [21, Prop. A.1.] to the radial case.

Proof of Proposition 1. We restrict our proof to the case $\beta \in (4/3, 2]$. Higher regularity, as well as the propagation, are treated in Appendix A.2. Fix $T > 0$. For $\omega \in \Omega$ and $t \in [0, T]$, define the Ornstein-Uhlenbeck process

$$Z(t, \omega) = \int_0^t S(t-s) dw_\phi(s), \quad t \geq 0, \quad \omega \in \Omega. \quad (2.5)$$

It is standard that since $\phi \in \mathbb{L}_2(H, V_\beta)$, Z is a random variable supported in the space $C([0, T]; V_\beta)$ – see [17]. Therefore we can take $z \in C([0, T]; V_\beta)$, and argue pathwise, considering the translated equation (1.10) with unknown v . For $h_0 \in V_\beta$, if a solution v exists up to $\tau = \tau(z) > 0$, it is well-known that $h := v + z$ gives a solution of (2.1) on $\{Z|_{[0, \tau]} = z|_{[0, \tau]}\}$. Thus, for each $z \in C([0, T]; V_\beta)$, we aim to find a fixed point v for the map $\Gamma = \Gamma_{h_0, z, T}$, defined as

$$\Gamma(v)(t) := S(t)h_0 + \int_0^t S(t-s)b(\cdot, v(s) + z(s)) ds, \quad \text{for } t \in [0, T]. \quad (2.6)$$

We will show that if $T_* > 0$ is sufficiently small, depending only on $\|z\|_{T, \beta}$ and $|h_0|_\beta$, then the mapping Γ is a contraction of a certain ball of $C([0, T_*]; V_\beta)$. It relies mainly on the three following properties, whose proof are given in Appendix A.1:

$$\left| (-A)^\alpha S(t) \right|_{\mathcal{L}(V_\beta)} \leq ct^{-\alpha}, \quad \text{for all } t > 0 \text{ and all } \alpha \in \mathbb{R}, \quad (2.7)$$

$$|b(\cdot, v)|_H \leq c'|v|_\beta^3, \quad \text{for all } v \in V_\beta, \quad (2.8)$$

$$|b(\cdot, u) - b(\cdot, v)|_H \leq c''|u - v|_\beta(|u|_\beta^2 + |v|_\beta^2) \quad \text{for all } u, v \in V_\beta, \quad (2.9)$$

with constants depending on $\alpha, \beta > 4/3$ and A .

Consider any z as above, and $h_0 \in V_\beta$. If $v \in C([0, T]; V_\beta)$, taking the V_β -norm in (2.6) and using (2.7) and (2.8) gives:

$$\|\Gamma(v)\|_{T, \beta} \leq |h_0|_\beta + c_1 T^{1-\beta/2} (\|v\|_{T, \beta}^3 + \|z\|_{T, \beta}^3). \quad (2.10)$$

Then, using (2.6), for $u, v \in C([0, T]; V_\beta)$, we have by (2.7) and (2.9):

$$\|\Gamma(u) - \Gamma(v)\|_{T, \beta} \leq c_2 T^{1-\beta/2} (\|u\|_{T, \beta}^2 + \|v\|_{T, \beta}^2 + 2\|z\|_{T, \beta}^2) \|u - v\|_{T, \beta}. \quad (2.11)$$

Set $R := |h_0|_\beta \vee \|z\|_{T, \beta} + 1$. Letting

$$T_* := \min \left(\frac{1}{4c_1 R^3}, \frac{1}{8c_2 R^2} \right)^{1/(1-\beta/2)}, \quad (2.12)$$

then (2.10) and (2.11) yield respectively $\|\Gamma(v)\|_{T_*, \beta} \leq R - 1/2$ and $\|\Gamma(u) - \Gamma(v)\|_{T_*, \beta} \leq (1/2)\|u - v\|_{T_*, \beta}$, so that:

- the ball $\mathbb{B}^R \subset C([0, T_*]; V_\beta)$ centered at 0 and of radius R is left invariant by Γ_{h_0, z, T_*} ;
- $\Gamma_{h_0, z, T_*} : \mathbb{B}^R \rightarrow \mathbb{B}^R$ is a contraction.

Applying Picard Theorem (the underlying space is complete), there exists a unique fixed point $v(h_0, z)$ for Γ_{h_0, z, T_*} , a mild solution to the perturbed equation (1.10), up to $t = T_*$. The maximal solution is obtained by reiteration of the argument. \blacksquare

For $h_0 \in V_\beta$, $z \in C([0, T_*]; V_\beta)$, if $R := |h_0|_\beta \vee \|z\|_{T, \beta} + 1$ and $T_*(R)$ is as in (2.12), then the unique fixed point of $\Gamma_{h_0, z, T_*}|_{\mathbb{B}^R}$, which we denote by v_0 , depends continuously on $z|_{[0, T_*]}$ and h_0 . Indeed, first note that by (2.12) we have

$$\|\Gamma_{h_1, \zeta}(v)\|_{T_*, \beta} \leq R - \frac{1}{4} \quad \text{and} \quad \|\Gamma_{h_1, \zeta}(u) - \Gamma_{h_1, \zeta}(v)\|_{T_*, \beta} \leq \frac{3}{4}\|u - v\|_{T_*, \beta}, \quad (2.13)$$

for (h_1, ζ) lying in some neighbourhood $\mathcal{V} \times \mathcal{W}$ of (h_0, z) . By the previous analysis, the bound (2.13) guarantees the existence the unique fixed point v_1 of Γ_{h_1, ζ, T_*} . For such (h_1, ζ) , using that $v_0 = \Gamma_{h_0, z}(v_0)$, $v_1 = \Gamma_{h_1, \zeta}(v_1)$, and re-using the properties (2.7)-(2.8)-(2.9), we immediately obtain

$$\|v_0 - v_1\|_{T_*, \beta} \leq |h_0 - h_1|_\beta + cT_*^{1-\beta/2}R^2\left(\|v_0 - v_1\|_{T_*, \beta} + \|z - \zeta\|_{T_*, \beta}\right),$$

so that the continuity of v at $(h_0, z) \in \mathcal{V} \times \mathcal{W}$ follows. This eventually gives the continuity for $h := v + z$, locally on $[0, T_*]$. The continuity of these functionals remains true up to the maximal times, as stated in the next lemma (the proof is done in Appendix A.2).

Lemma 1 (Continuous dependence). *Let $T > 0$, $z \in C([0, T]; V_\beta)$, $h_0 \in V_\beta$ and assume that $h(h_0, z, \cdot)$ exists on $[0, T]$. There exist open sets $\mathcal{V} \subset V_\beta$ and $\mathcal{W} \subset C([0, T]; V_\beta)$, with $(h_0, z) \in \mathcal{V} \times \mathcal{W}$, such that for all $(h_1, \zeta) \in \mathcal{V} \times \mathcal{W}$, there exists a unique mild solution $h(h_1, \zeta, \cdot) \in C([0, T]; V_\beta)$ of (2.1).*

Moreover, the mapping $\mathcal{V} \times \mathcal{W} \rightarrow C([0, T]; V_\beta)$, $(h_1, \zeta) \mapsto h(h_1, \zeta, \cdot)|_{[0, T]}$, is continuous.

2.2 Finite-time blow-up

In the sequel, when $(h_0, z) \in V_\beta \times C([0, \infty); V_\beta)$, we will systematically denote by $(h(h_0, z), \tau^\beta(h_0, z))$ the mild solution of (2.1) on $\{Z = z\}$, and its maximal time of existence in V_β , namely:

$$\left[\begin{array}{l} h(h_0, z) := v + z, \\ \text{where } v = v(h_0, z) \text{ solves in the mild sense:} \\ \quad \left\{ \begin{array}{l} \partial_t v = Av + b(\cdot, v + z) \text{ on } [0, \tau^\beta(h_0, z)) \times I, \\ v|_{t=0} = h_0, \end{array} \right. \\ \text{and where } \tau^\beta(h_0, z) < \infty \text{ implies } \limsup_{t \rightarrow \tau^\beta(h_0, z)} |h(h_0, z, t)|_{V_\beta} = \infty. \end{array} \right. \quad (2.14)$$

The main argument in the proof that blow-up happens with positive probability, for any initial data, is the following lemma. Its proof will be given in Section 3.

Lemma 2 (main lemma). *Let $\beta > 2$, and $\phi \in \mathbb{L}_2(H, V_\beta)$. Then, for any $t_* > 0$, there exist two subsets \mathfrak{H} of V_β , \mathfrak{Z} of $C([0, t_*]; V_\beta)$, with nonempty interiors, such that for all $(h_0, z) \in \mathfrak{H} \times \mathfrak{Z}$:*

$$\tau^\beta(h_0, z) \leq t_*.$$

Now, given $T_1 > 0$ and $h_0, h_1 \in V_\beta$, observe that there exists a control $z_1 \in C([0, T_1]; V_\beta)$ with $z_1(0) = 0$, such that: $h(h_0, z_1, \cdot)$ exists on $[0, T_1]$, and

$$h(h_0, z_1, T_1) = h_1. \quad (2.15)$$

Indeed, similar to [18], we set for $t \in [0, T_1]$: $\varphi(t) := \frac{T_1-t}{T_1} h_0 + \frac{t}{T_1} h_1$, and define $f(t) := (\varphi(t) - h_0) - \int_0^t (A\varphi(s) + b(\cdot, \varphi)(s)) ds$. Taking now

$$z_1(t) = \int_0^t S(t-s) \frac{df}{ds} ds, \quad t \in [0, T_1],$$

then the map $V := \varphi - z_1$ is a solution of the translated equation (1.10) with $z = z_1$, so that by the uniqueness part above there holds: $\varphi = h(h_0, z_1, \cdot)|_{[0, T_1]}$. Note that $\frac{df}{dt} \in C([0, T_1]; V_{\beta-2})$, so that by classical theory of parabolic equations, we have indeed $z_1 \in C([0, T_1]; V_\beta)$.

End of the proof of Theorem 1. Fix $t_* > 0$, $s \in (0, t_*)$ and take $\mathfrak{H}, \mathfrak{Z}$ as in Lemma 2, with t_* replaced $t_* - s$. Since it is nonempty, we may consider an element h_1 in the interior of \mathfrak{H} . By the controlability property (2.15), there exists $z_1 \in C([0, s]; V_\beta)$ such that $h(h_0, z_1, \cdot)$ is defined on $[0, s]$ and $h(h_0, z_1, s) = h_1$. Using in addition Lemma 1, we see that there exists a neighbourhood \mathcal{V}_1 of z_1 in $C([0, s]; V_\beta)$, such that

$$\forall z \in \mathcal{V}_1, \quad h(h_0, z, s) \in \mathfrak{H}.$$

Since $\ker \phi^* = \{0\}$, then ϕ has dense range in V_β and the process $Z(t) = \int_0^t S(t-\sigma) dw_\phi(\sigma)$, $t \geq 0$, is non degenerate. Therefore,

$$p_0 := \mathbb{P} \circ Z|_{[0, s]}^{-1}(\mathcal{V}_1) > 0, \quad (2.16)$$

and similarly

$$p_1 := \mathbb{P} \circ Z|_{[0, t_*-s]}^{-1}(\mathfrak{Z}) > 0, \quad (2.17)$$

Now, define the extended state space $\mathfrak{X} = V_\beta \cup \{\Delta\}$ where the terminal state Δ is an isolated point, and extend the process $X_{t, h_0}(\omega) := h(h_0, t, Z(\omega))$ on \mathfrak{X} , by achieving Δ if and only if $t \geq \tau^\beta(h_0, Z(\omega))$. By standard arguments (see e.g. [40] and references therein), the family of probability measures $(\mathbb{P}_x \equiv \mathbb{P} \circ X_{\cdot, x}^{-1})_{x \in \mathfrak{X}}$ on $\bar{W} := C([0, \infty); \mathfrak{X})$, the space of trajectories equipped with the σ -algebra corresponding to Borelian sets, is Markovian. Letting $A := \{w \in \bar{W} : \tau(w) \geq s\}$, we have

$$\mathbb{P}_x(A \cap \{w : w(t_*) = \Delta\}) = \int_A \mathbb{P}_{w'(s)}(w : w(t_* - s) = \Delta) \mathbb{P}_x(dw'), \quad (2.18)$$

Denote by $P(x, t; \cdot)$ then associated transition probabilities, namely $\mathbb{P}_x(w : w(t) \in \Gamma)$ where $\Gamma \subset \mathfrak{X}$ is Borelian, and by $\pi_s : \bar{W} \rightarrow \mathfrak{X}$, $w \mapsto w(s)$. Then (2.18) implies

$$\mathbb{P}_x(\tau \leq t_*) \geq \int_{\mathfrak{H} \cap \pi_s(A)} P(t_* - s, \xi; \{\Delta\}) P(s, x; d\xi) \geq p_1 \int_{\mathfrak{H} \cap \pi_s(A)} P(h_0, s; d\xi),$$

where we have used (2.17) to bound $P(t_* - s, \xi, \{\Delta\})$ independently of $\xi \in \mathfrak{H}$. Using in addition (2.16), we obtain $\mathbb{P}_x(\tau \leq t_*) > p_1 p_0$ which is positive, and by Remark 2.1 Theorem 1 is proved. \blacksquare

3 Proof of Lemma 2

3.1 Preliminary material

As in Chang-Ding-Ye's proof, we use a comparison principle for the scalar parabolic equation (1.3). It is however different from that of [15, 9], because the nonlinearity depends on the realization of the Ornstein Uhlenbeck process $Z : \Omega \rightarrow C([0, T]; V_\beta)$. However additiveness of the noise in (1.9) allows to appeal to deterministic theory only, fixing $\omega \in \Omega$ and letting $z := Z(\omega)$.

We consider equations of the form

$$\partial_t f = Af - \frac{p(z(t, r) + f(t, r))}{r^2}, \quad \text{for } (t, r) \in [0, \kappa] \times I \setminus \{0\}, \quad (3.1)$$

where $p : \mathbb{R} \rightarrow \mathbb{R}$ vanishes at the origin and

$$z \in C([0, \kappa]; V_\beta) \text{ for some } \beta > 1 \text{ and } z|_{\{0\} \times I} = 0 \quad (3.2)$$

(note that for such z we have in fact $z|_\Sigma = 0$). In order to take into account the main cases we have in mind, we will assume that the nonlinearity fulfills the following properties.

Assumptions on p . We will assume that $p : \mathbb{R} \rightarrow \mathbb{R}$ is of class C^2 around the origin, and that

$$p(0) = 0; \quad p'(0) > -1; \quad |p(x) - p(y)| \leq K|x - y|, \quad \forall x, y \in \mathbb{R}, \quad (3.3)$$

for some universal constant $K > 0$.

Remark 3.1. Assumptions (3.3) do cover the cases where: (a) $p(x) = 0$ (comparison principle for the linear equation $(\partial_t - A)f = 0$), and (b) $p(x) = \sin(2x)/2 - x$ (comparison principle for (2.1)).

The proof of the following result is postponed at the Appendix.

Comparison principle for (3.1). Fixing some $\beta > 1$, assume that the assumptions (3.2) and (3.3) are fulfilled, and that we are given $f, g \in C([0, \kappa]; V_\beta) \cap C^1([0, \kappa]; H)$, such that

$$\begin{aligned} \text{(i)} \quad & -\int_0^\kappa \int_J f \partial_t \zeta r \, dr \, dt \leq -\int_0^\kappa \int_J (\partial_r f \partial_r \zeta + \frac{f + p(z+f)}{r^2} \zeta) r \, dr \, dt, \\ \text{(ii)} \quad & -\int_0^\kappa \int_J g \partial_t \zeta r \, dr \, dt \geq -\int_0^\kappa \int_J (\partial_r g \partial_r \zeta + \frac{g + p(z+g)}{r^2} \zeta) r \, dr \, dt, \end{aligned}$$

for all nonnegative $\zeta \in C^\infty([0, \kappa] \times J)$ with $\zeta(t, r) = 0$ on the parabolic boundary $\Sigma := \{0\} \times J \cup [0, \kappa] \times \partial J$. Assume moreover that $f \leq g$ on Σ . Then:

For almost every $(t, r) \in [0, \kappa] \times J$, we have $f(t, r) \leq g(t, r)$.

In the sequel, for $k > 0$, and $r \in I$, we denote by

$$\chi_k(r) := k(2r - 3r^3 + r^5) = kr(1 - r^2)(2 - r^2). \quad (3.4)$$

see Fig. 2. Initial data h_0 will be compared according to their position with respect to the reference family $(\chi_k)_{k \in \mathbb{N}}$, see subsections below. The choice of such parabolae rules out the pathological case where $\tau^\beta(\chi_k, z)$ equals 0. Indeed, on the one hand we have $A^2 \chi_k(r) = 48kr$, $r \in I \setminus \{0\}$, which belongs to $V_{1/2-\epsilon}$ for any $\epsilon > 0$ (this fact is left to the reader). On the other hand: $\chi_k(\partial I) \equiv 0$, and $A\chi_k(\partial I) \equiv 0$, which ensures that for $k \in \mathbb{R}$:

$$\chi_k \in V_\alpha, \quad \text{for any } \alpha < \frac{9}{2}. \quad (3.5)$$

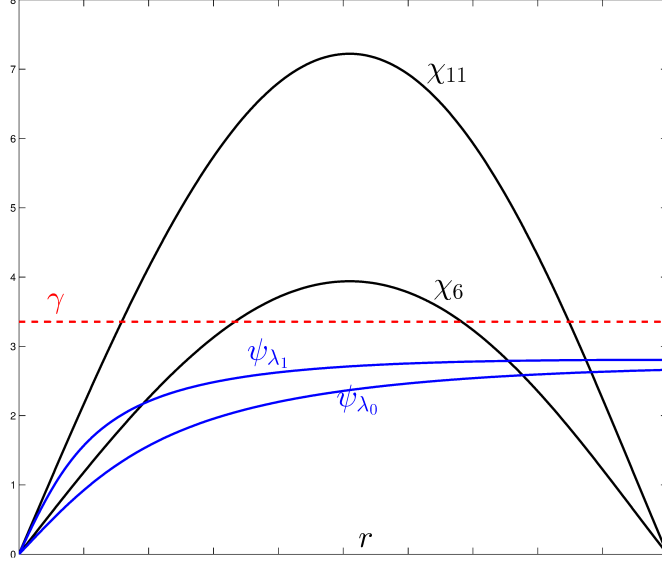


Figure 2: Plots of $\psi_{\lambda_0}, \psi_{\lambda_1}$ for some $\lambda_1 < \lambda_0$, together with χ_6, χ_{11} and γ , for $r \in I \equiv [0, 1]$.

3.2 Blow-up for a fixed trajectory

Our approach in the proof of Lemma 2 is to show first that given $\beta > 2$, and a fixed $z \in C([0, 2t_*]; V_\beta)$ with $z(0) = 0$, there exists a map $\chi \in V_\beta$ (depending on z) such that for every h_0 lying everywhere over χ , the associated solution $h(h_0, z, \cdot)$ blows up before t_* . The proof will be completed in the next subsection, using a topological argument.

Claim 1. Let $\beta > 2$, and fix $t_* > 0$. There exists $\bar{\eta} > 0$, such that for all $z \in C([0, 2t_*]; V_\beta)$ with $\|z\|_{2t_*, \beta} \leq \bar{\eta}$, there exists a parabola $\chi_* = \chi_*(z)$ belonging to the family (3.4), and satisfying the property that: if $h_0 \in V_\beta$ with $h_0 \geq \chi_*$, then

$$\tau^\beta(h_0, z) \leq t_*.$$

Moreover, the pre-blow-up set $\mathcal{H} = \{h_0 \in V_\beta, h_0 \geq \chi_*\}$, has *nonempty interior* in V_β .

The proof of Claim 1 will be done in several steps. In the following lemma, we exhibit an explicit family of maps $\{\psi_{\epsilon, \mu, \lambda_0, \xi}\}$ satisfying the differential inequality

$$\partial_t \psi \leq A\psi + b(r, \psi + z), \quad (3.6)$$

up to some positive time. Since in the sequel $\beta > 1$, we restrict our attention to maps that have a continuous version (see Remark 1.3), and when $f, g \in V_\beta$ we write $f \leq g$ for $\langle f, \zeta \rangle \leq \langle g, \zeta \rangle$ for all non-negative $\zeta \in C_0^\infty(I)$.

In the following Lemma we denote by $J := [0, r_1]$ with some $r_1 \in (0, 1)$.

Lemma 3. Fix z in $C([0, \infty); V_\beta)$ with $z(0) = 0$. For $\lambda_0, \epsilon, \delta > 0$, define $\lambda = \lambda_{\epsilon, \delta, \lambda_0} : t \in [0, T_{\lambda_0}) \mapsto \lambda(t)$ as the solution of the ODE :

$$\lambda' = -\delta \lambda^\epsilon, \quad 0 < t \leq T_{\lambda_0} := \frac{\lambda_0^{1-\epsilon}}{(1-\epsilon)\delta}, \quad \text{with initial data } \lambda(0) = \lambda_0. \quad (3.7)$$

Assume that there exist $t_+ > 0$, $\xi \in V_\beta$, $\xi \geq 0$, depending on z , such that

$$x(t, r) := S(t)\xi(r) + z(t, r) \geq 0 \text{ for } t \in [0, t_+] \text{ and } r \in J, \quad (3.8)$$

where $S(t) = e^{tA}$, see (1.12).

Fix $0 < \epsilon < 1$. There exist positive constants $\bar{\mu}(\epsilon)$, $\bar{\delta}(\epsilon)$, such that for all $\mu \geq \bar{\mu}(\epsilon)$ and $0 < \delta \leq \bar{\delta}(\epsilon)$, for all $\lambda_0 > 0$ defining $\lambda = \lambda_{\epsilon, \delta, \lambda_0}(t)$ as in (3.7), then the map

$$\psi(r, t) = \arccos\left(\frac{\lambda(t)^2 - r^2}{\lambda(t)^2 + r^2}\right) + \arccos\left(\frac{\mu^2 - r^{2+2\epsilon}}{\mu^2 + r^{2+2\epsilon}}\right) + S(t)\xi(r), \quad (3.9)$$

fulfills the differential inequality (3.6) on $[0, t_+ \wedge T_{\lambda_0}] \times J$.

Proof of Lemma 3. Let $0 < \epsilon < 1$. As in [15], we set for $(\lambda, r) \in \mathbb{R}_+^* \times J$:

$$\varphi_\lambda(r) := \arccos\left(\frac{\lambda^2 - r^2}{\lambda^2 + r^2}\right), \quad \theta_{\epsilon, \mu}(r) := \arccos\left(\frac{\mu^2 - r^{2+2\epsilon}}{\mu^2 + r^{2+2\epsilon}}\right). \quad (3.10)$$

Recall that for any fixed triplet $\lambda, \epsilon, \mu > 0$, the maps $\varphi_\lambda, \theta_{\epsilon, \mu}$ satisfy for $r \in J$ (see [15]):

$$\begin{aligned} A\varphi_\lambda(r) &= \frac{\sin 2\varphi_\lambda(r) - 2\varphi_\lambda(r)}{2r^2} \\ A\theta_{\epsilon, \mu}(r) &= \frac{(1 + \epsilon)^2 \sin 2\theta_{\epsilon, \mu}(r) - 2\theta_{\epsilon, \mu}(r)}{2r^2} \end{aligned} \quad (3.11)$$

Now, since $\theta_{\epsilon, \mu}(r) \rightarrow 0$ as $\mu \rightarrow \infty$, it is possible to choose a parameter $\bar{\mu}(\epsilon)$, such that for all $r \in J$,

$$\cos \theta_{\epsilon, \mu}(r) \geq \frac{1}{1 + \epsilon}. \quad (3.12)$$

Take $\mu \geq \bar{\mu}$, and let $z \in C([0, t_+]; V_\beta)$ be such that $x = S\xi + z$ takes nonnegative values on $[0, t_+] \times J$. For $t \in [0, T_{\lambda_0})$, $r \in J$, define $\psi(t, r) := \varphi_{\lambda(t)}(r) + \theta(r) + S(t)\xi(r)$, and denote by $\theta := \theta_{\epsilon, \mu}(\cdot)$, $\varphi := \varphi_{\lambda(\cdot)}(\cdot)$, and $S\xi := t \in \mathbb{R}_+ \mapsto S(t)\xi$. On the one side, using (3.11), the trigonometric identities $\sin 2\theta = 2 \cos \theta \sin \theta$ and $\sin 2\varphi - \sin 2(\varphi + \theta) = -2 \sin \theta \cos(2\varphi + \theta)$, there comes

$$\begin{aligned} A\psi + b(\cdot, \psi + z) &= A(\varphi + \theta + S\xi) + b(\cdot, \varphi + \theta + x) \\ &= \frac{1}{2r^2} [(1 + \epsilon)^2 \sin 2\theta + \sin 2\varphi - \sin 2(\varphi + \theta) \\ &\quad + 2x + \sin 2(\varphi + \theta) - \sin 2(\varphi + \theta + x)] + AS\xi \\ &= \frac{1}{2r^2} [2(1 + \epsilon)^2 \sin \theta \cos \theta - 2 \sin \theta \cos(2\varphi + \theta) \\ &\quad + F_{\varphi, \theta}(x)] + AS\xi \end{aligned} \quad (3.13)$$

where we denote by $F_{\varphi, \theta}(x) = 2x - (\sin 2(\varphi + \theta + x) - \sin 2(\varphi + \theta))$, $x \in \mathbb{R}$, for $\varphi, \theta \in \mathbb{R}$. Using (3.12), the right hand side in (3.13) is bounded below by $1/(2r^2)[(2 + 2\epsilon - 2 \cos(2\varphi + \theta)) \sin \theta + F_{\varphi, \theta}(x)] + AS\xi$, so that $A\psi + b(r, \psi + z) \geq 1/(2r^2)[2\epsilon \sin \theta + F_{\varphi, \theta}(x)] + AS\xi$. Expanding $\sin \theta$, we eventually obtain

$$\begin{aligned} A\psi + b(r, \psi + z)(t, r) &\geq \frac{2\epsilon \mu r^{\epsilon-1}}{\mu^2 + r^{2(1+\epsilon)}} + \frac{F_{\varphi, \theta}(x(t, r))}{r^2} + AS(t)\xi(r) \\ &\geq \frac{2\epsilon \mu r^{\epsilon-1}}{\mu^2 + 1} + \frac{F_{\varphi, \theta}(x(t, r))}{r^2} + AS(t)\xi(r), \end{aligned}$$

a.e. on $[0, t_+] \times J \setminus \{0\}$ (and therefore in the sense of positive test functions). Now, regardless of the values taken by the parameters θ, φ , the map $F_{\varphi, \theta}$ vanishes at the origin, and has nonnegative derivative on \mathbb{R}_+ . We deduce that since $x \geq 0$ on $[0, t_+] \times J$, then so is $F_{\varphi, \theta}(x)$. Moreover, simple computations show that for $r \in J$:

$$\partial_t \psi(t, r) = \frac{2\delta \lambda(t)^\epsilon r}{\lambda(t)^2 + r^2} + AS(t)\xi(r) ,$$

Thus, if for every (t, r) in $[0, t_+] \times J$, we have $2\epsilon \mu r^{\epsilon-1}/(\mu^2 + 1) \geq 2\delta \lambda(t)^\epsilon r/(\lambda(t)^2 + r^2)$, then $\partial_t \psi \leq A\psi + b(r, \psi + z)$ holds. Setting $s = r/\lambda(t)$, it is however sufficient to verify that: $\sup_{s \in \mathbb{R}} \frac{s^{2-\epsilon}}{1+s^2} \leq \frac{\mu\epsilon}{\delta(\mu^2+1)}$, which is true if $\delta \geq \bar{\delta}$ for $\bar{\mu} > 0$ as in (3.12). This proves the lemma. \blacksquare

Remark 3.2. Let $\beta > 2$, and consider ξ, z, t_+ as in Lemma 3, and assume that

$$t_+ \geq T_{\lambda_0} \equiv \frac{\lambda_0^{1-\epsilon}}{(1-\epsilon)\delta} . \quad (3.14)$$

Then, the map $f := \psi_{\epsilon, \mu, \lambda_0, \xi}$ constructed above blows up at $t = T_{\lambda_0}$. Indeed, since $\xi \in V_\beta$ with $\beta > 2$, then $|\partial_r S(t)\xi|_{L^\infty}$ is bounded for $t \in [0, T_{\lambda_0})$ (see Remark 1.3), and:

$$\partial_r \psi(t, 0) = \partial_r \varphi(t, 0) + \theta'(0) + \partial_r (S(t)\xi)(0) = \frac{2}{\lambda(t)} + \partial_r (S(t)\xi)(0) \xrightarrow{t \rightarrow T_{\lambda_0}} \infty .$$

Let $(h, \tau) = (h(h_0, z), \tau^\beta(h_0, z))$ be the mild solution defined in (2.14), where $h_0 \in V_\beta$. Assume $\tau \geq T_{\lambda_0}$, and that:

$$f \leq g \quad \text{on } \{0\} \times J \cup [0, T_{\lambda_0}) \times \partial J, \quad (3.15)$$

where we let $g := h - z$, and $J := [0, r_1]$ for some $r_1 \in (0, 1)$.

Up to some straightforward extension on the whole interval I , observe that $f \in C([0, T]; V_\beta) \cap C^1([0, T]; H)$, as can be shown by direct computations. On the other hand, recall that $g \equiv h(z) - z$ satisfies $g(t) = h_0 + \int_0^t S(t-s)b(\cdot, g+z)ds$ for $t \leq \tau$. Since $\beta > 4/3$, then (2.8) yields $b(\cdot, g+z) \in C([0, T]; H)$, so that

$$g \in C([0, T]; V_\beta) \cap C^1([0, T]; H) . \quad (3.16)$$

By (3.15), (3.16) and Lemma 3, the comparison principle for (1.10) can be applied so that $f \leq g$ on $[0, T_{\lambda_0}) \times J$. Since the maps f, g vanish at the origin regardless of the time variable, it follows that:

$$\partial_r f(t, 0) \leq \partial_r g(t, 0), \quad \text{for all } t \in [0, T_{\lambda_0}) ,$$

and then $|\partial_r h|_{L^\infty} \rightarrow \infty$ as $t \rightarrow T_{\lambda_0}$, which by Remark 1.3 implies blow-up also in the sense that $\limsup_{t \rightarrow T_{\lambda_0}} |h(t)|_\beta = \infty$.

We can now turn to the proof of Claim 1.

Proof of Claim 1. Fix $2 < \beta \leq 4$. For each $z \in C([0, 2t_*]; V_\beta)$, and $\xi \in V_\beta$, we define $x = x_{\xi, z}$ by:

$$x(t) = S(t)\xi + z(t), \quad \text{for } t \leq 2t_* . \quad (3.17)$$

In what follows we denote by J the compact interval $[0, 1/2]$.

Step 1: nonnegativeness of x up to a positive time. Assume that $\xi \geq \chi_1$ on J , where χ_1 is the parabola defined by (3.4). Note that such $\xi \in V_\beta$ exists for $\beta > 2$ since it suffices to let for instance $\xi := \chi_1$, see (3.5). Our aim now is to show that if the perturbation z is not too large in $C([0, 2t_*]; V_\beta)$, then the map x defined above stays nonnegative on J .

We first claim that there exists a constant $\eta > 0$, such that for all $\xi, y \in V_\beta$ with $\xi \geq \chi_1$,

$$|\xi - y|_\beta \leq 2\eta \Rightarrow |y|_J \geq 0. \quad (3.18)$$

Indeed, since $\beta > 2$, then there exists $c_\beta > 0$, such that for all $y \in V_\beta$, (see Remark 1.3),

$$|\partial_r \xi - \partial_r y|_{L^\infty(J)} \leq c_\beta |\xi - y|_\beta.$$

Choose $\eta = c/(2c_\beta)$, where c is such that $\chi_1(r) - cr \geq 0$ for $r \in J$ (note that c and therefore η do not depend on ξ), so that $|y - \xi|_\beta \leq \eta$ will imply $|\partial_r y - \partial_r \xi|_{L^\infty} \leq c/2$. We conclude by the Mean Value Theorem, observing first that both maps equal zero at the origin: if $|\xi - y|_\beta \leq \eta$, then $\forall r \in J$, $y(r) \geq \xi(r) - cr \geq \chi_1(r) - cr$ and thus $y(r) \geq 0$, which proves (3.18).

Furthermore, for a fixed $\xi \in V_\beta$ with $\xi \geq \chi_1$, since S is a strongly continuous semi-group, there exists $t_+(\xi) > 0$ such that

$$\text{for all } t \in [0, t_+(\xi)], \quad |S(t)\xi - \xi|_\beta \leq \eta,$$

and thus for $0 \leq t \leq t_+(\xi)$, $\|z\|_{2t_*, \beta} \leq \eta$, the map x defined in (3.17) verifies

$$|x(t) - \xi|_\beta \leq |S(t)\xi - \xi|_\beta + |z(t)|_\beta \leq 2\eta. \quad (3.19)$$

We have to get rid of the dependence of $t_+(\xi)$ with respect to ξ . But if $\xi \in V_\beta$ with $\xi \geq \chi_1$ on I , apply the linear comparison principle (see the previous subsection) on the whole interval I to $f := S(\cdot)\chi_1$, $g := S(\cdot)\xi$, $\kappa := 2t_*$ (note that we have $f \leq g$ on $\{0\} \times I \cup [0, 2t_*] \times \partial I$). We obtain that

$$t_+(\xi) \geq t_+(\chi_1).$$

Now define $t_+ := t_+(\chi_1)$. We have proven that there exists $\eta > 0$ such that for all $t \in [0, t_+]$, for all $\xi \in V_\beta$ with $\xi \geq \chi_1$ on I , for all $z \in C([0, 2t_*]; V_\beta)$ with $\|z\|_{2t_*, \beta} \leq \eta$, then

$$x|_{[0, t_*] \times J} \geq 0. \quad (3.20)$$

Step 2. Construction of a pre-blow-up set for a fixed z . Once and for all, fix η as in Step 1, $z \in C([0, 2t_*]; V_\beta)$ with $\|z\|_{2t_*, \beta} \leq \eta$, and $\xi \in V_\beta$ with $\xi \geq \chi_1$ on J , so that (3.20) holds for $x = x_{\xi, z}$.

It suffices to prove the proposition with $t_* \wedge t_+$ instead of t_* . Therefore, without loss of generality we assume in the sequel that

$$t_+ = t_*.$$

In order to lighten the notations we also denote by $\tau = \tau^\beta(\cdot, z)$, and $h = h(\cdot, z)$. Take any $0 < \epsilon < 1$, and fix $\mu \geq \bar{\mu}(\epsilon)$, $\delta \leq \bar{\delta}(\epsilon)$ and $\lambda = \lambda_{\epsilon, \delta, \lambda_0}(t)$ as in Lemma 3, where $\lambda_0 > 0$ is chosen such that

$$T_{\lambda_0} = \frac{\lambda_0^{1-\epsilon}}{\delta(1-\epsilon)} \leq t_+,$$

so that we know by Lemma 3, that the map $f_0 := \psi_{\epsilon, \mu, \lambda_0, \xi}$ defined as in (3.9), fulfills

$$\partial_t f_0 \leq A f_0 + b(r, f_0 + z) \text{ on } [0, T_{\lambda_0}) \times J, \quad (3.21)$$

with blow-up at $t = T_{\lambda_0}$. Our strategy is to take $h_0 \geq f_0|_{t=0}$, compare $g := h(h_0, z, \cdot) - z$ with this ansatz, and then conclude by Remark 3.2 that blow-up of h happens before t_+ . For that purpose it remains however to chose h_0 in such a way (3.15) holds. Note that if $h_0 \in V_\beta$ is taken such that

$$(h(h_0) - z)|_{[0, t_+] \times \{\frac{1}{2}\}} > \sup_{(r, \lambda) \in J \times \mathbb{R}_+^*} \left(\varphi_\lambda(r) + \theta(r) + S(t)\xi(r) \right), \quad (3.22)$$

then $(h(h_0) - z)|_{[0, t_+] \times \{1/2\}} \geq \psi_{\epsilon, \mu, \lambda_0, \xi}(t, 1/2)$, with ψ as in (3.9), regardless of ϵ, μ, λ_0 , and $0 \leq t \leq t_+$. In particular (3.22) will imply the bound needed on $[0, t_+] \times \partial J$. Moreover, note that π is an upper bound for the family of maps $(\varphi_\lambda(\cdot))_{\lambda > 0}$ (see Figure 2). This motivates the following definition: let

$$\gamma := \pi + |\theta_{\epsilon, \mu}|_{L^\infty} + \sup_{t \geq 0} |S(t)\xi|_{L^\infty}, \quad (3.23)$$

and for $h_0 \in V_\beta$, define

$$t_\Sigma(h_0) = \inf \left\{ 0 \leq t \leq \tau(h_0), (h(h_0, t) - z(t))|_{\{\frac{1}{2}\}} \leq \gamma \right\}, \quad (3.24)$$

with the understanding that $t_\Sigma(h_0) = \tau(h_0)$ if the set is empty.

Note that γ is well-defined. Indeed for any $u = \sum_k u_k e_k \in V_\beta$, by Remark 1.3, since $\beta > 1$, the mapping $t \mapsto |S(t)u|_{L^\infty} = |\sum_k u_k e^{t\lambda_k} e_k|_{L^\infty}$, $t \geq 0$, is bounded (for a thorough description of the eigenpairs (λ_k, e_k) the reader might refer to the appendix).

We claim now that there exists an integer $k = k(z) \geq 1$ such that for all $h_0 \in V_\beta$, if $h_0 \geq \chi_k$ on I , then

$$\tau(h_0) \leq t_+. \quad (3.25)$$

Indeed, let $h_0 \in V_\beta$ with $h_0|_J \geq f_0|_{\{0\} \times J}$ and assume that $\tau(h_0) > t_+$. Note that necessarily $\inf_{t \in [0, t_+]} (h(h_0) - z)|_{[0, t_+] \times \{\frac{1}{2}\}} < \gamma$, otherwise by comparison between f_0 and $g := h(h_0) - z$, Remark 3.2 would yield blow-up for $h(h_0)$ before t_+ . So we have

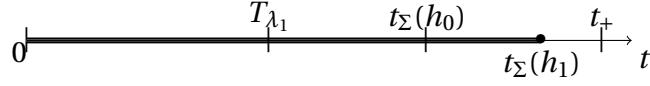
$$t_\Sigma(h_0) \leq t_+. \quad (3.26)$$

Now, choose any $\lambda_1 > 0$ with $T_{\lambda_1} = \lambda_1^{1-\epsilon} / (\delta(1-\epsilon)) \leq t_\Sigma(h_0)$, and define $f_1 := \psi_{\epsilon, \mu, \lambda_1, \xi}$ by the formula (3.9) with λ_1 instead of λ_0 . Since arccos is Lipschitz out of 0, we can always find $k \geq 1$ such that for $r \in J$:

$$\chi_k(r) \geq f_1(0, r) \equiv \arccos\left(\frac{\lambda_1^2 - r^2}{\lambda_1^2 + r^2}\right) + \arccos\left(\frac{\mu^2 - r^{2+2\epsilon}}{\mu^2 + r^{2+2\epsilon}}\right) + \xi(r), \quad (3.27)$$

where χ_k is as in (3.4). Consider any $h_1 \in V_\beta$ with $h_1 \geq \chi_k$ on I . One has the following alternative.

- *Case 1:* $t_\Sigma(h_1) \geq t_\Sigma(h_0)$.

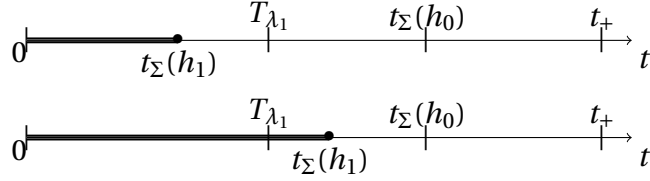


In this situation, we have:

$$T_{\lambda_1} \leq t_{\Sigma}(h_1) \leq \tau(h_1), \text{ and } h(h_1) \geq f_1 \text{ on } \{0\} \times J \cup [0, T_{\lambda_1}] \times \partial J, \quad (3.28)$$

and the comparison principle for (1.10) can be applied on the thick part of the above line segment, in particular with $\kappa = T_{\lambda_1}$, $f := f_1$ and $g := h(h_1) - z$. By Remark 3.2 we obtain that h blows-up before T_{λ_1} , whence $T_{\lambda_1} = \tau(h_1) \leq t_+$.

- *Case 2:* $t_{\Sigma}(h_1) < t_{\Sigma}(h_0)$.



In this case, apply the comparison principle for (1.10) on the whole interval I with $\kappa := \tau(h_0) \wedge \tau(h_1)$, $f := h(h_0) - z$, and $g := h(h_1) - z$, so that in particular:

$$\text{on } [0, \tau(h_0) \wedge \tau(h_1)), \text{ there holds } f\left(\cdot, \frac{1}{2}\right) \leq g\left(\cdot, \frac{1}{2}\right). \quad (3.29)$$

Therefore, in this case one has necessarily $\tau(h_1) = t_{\Sigma}(h_1)$, otherwise we would have

$$g\left(t_{\Sigma}(h_1), \frac{1}{2}\right) \equiv \gamma < f\left(t_{\Sigma}(h_1), \frac{1}{2}\right),$$

contradicting (3.29). Moreover, one has $t_{\Sigma}(h_1) \leq t_+$, and thus $\tau(h_1) \leq t_+$.

We see that in both cases (3.25) is true, and the claim implies that

$$\mathfrak{H} := \{h_1 \in V_{\beta}, \ h_1 \geq \chi_{k(z)}\}$$

defines a pre-blow-up set for the individual element z .

Step 3. Nonemptiness of \mathfrak{H} . It suffices to show the result when $k = 1$, namely that the set $\mathfrak{H} = \{h_1 \in V_{\beta}, \ h_1 \geq \chi_1\}$ has nonempty interior for the topology of V_{β} . Set $h_0 = \chi_2 \in \mathfrak{H}$, so that $h_0 \in \mathfrak{H}$. By Remark 1.3, since $\beta > 2$, there exists a sufficiently small radius $R > 0$ such that if $h_1 \in V_{\beta}$ with $|h_1 - h_0|_{\beta}$, then $|\partial_r h_1 - \partial_r h_0|_{L^{\infty}} \leq 1/2$. By the Mean Value Theorem, since h_0 and h_1 vanish for $r \in \{0, 1\}$, then for all $r \in [0, 1/2]$: $|h_1(r) - h_0(r)| \leq (1/2)r \leq r(1 - r)$, and the same holds when $r \in [1/2, 1]$. Thus for a.e. $r \in I$:

$$|h_1(r) - h_0(r)| \leq r(1 - r).$$

The reader may also check that

$$\forall r \in I, \ \chi_1(r) = r(1 - r^2)(2 - r^2) \geq r(1 - r).$$

Thus, for all h_1 belonging to an open ball centered at $h_0 = \chi_2$, and for all $r \in I$: $h_1(r) \geq \chi_2(r) - cr(1 - r) \geq \chi_1(r)$, which means that $h_1 \in \mathfrak{H}$. This finishes the proof of Claim 1. ■

3.3 Globalization, conclusion and closing remarks

Let $4 \geq \beta > 2$. Let $\bar{\eta} > 0$ taken as in Claim 1. So far, we have shown that given a trajectory z in the ball $\mathbb{B} \subset C([0, 2t_*]; V_\beta)$, centered at zero and of radius $\bar{\eta}$, there exist an integer $k(z)$, such that for all $h_0 \in V_\beta$, $h_0 \geq \chi_{k(z)}$, then $\tau^\beta(h_0, z) \leq t_*$. To conclude we need some globalization technique to reverse the quantifiers. Define

$$\mathfrak{F}_k(t_*) = \{z \in \mathbb{B}, \forall h_0 \in V_\beta \text{ with } h_0 \geq \chi_k \text{ on } I, \tau_z^\beta(h_0) \leq t_*\}.$$

We claim that $\mathfrak{F}_k(t_*)$ is a closed subset of \mathbb{B} . Indeed, by definition: if $z \in \mathbb{B} \setminus \mathfrak{F}_k(t_*)$, there exists $h_0 \in V_\beta$ with $h_0 \geq \chi_k$ on I and $\tau(h_0, z) > t_*$. Let $z^n \in \mathbb{B} \rightarrow z$ in \mathbb{B} , as $n \rightarrow \infty$. Let $\epsilon > 0$ such that $h(h_0, z, \cdot)$ is defined on $[0, t_* + \epsilon]$. By Lemma 1, $h(h_0, z^n, \cdot)$ will be defined up to $t_* + \epsilon$, provided n is large enough. And thus $(\mathfrak{F}_k(t_*))^c$ is an open set of \mathbb{B} , which proves the claim.

By Claim 1, if $z \in \mathbb{B}$, then there exists k such that $z \in \mathfrak{F}_k(t_*)$, thus

$$\mathbb{B} = \bigcup_{k \in \mathbb{N}} \mathfrak{F}_k(t_*).$$

Hence, by Baire's Theorem, there exists at least one k^* such that $\mathfrak{F}_{k^*}(t_*)$ has non-empty interior. Thus we can set $\mathfrak{J} = \mathfrak{F}_{k^*}(t_*)$. If we define $\mathfrak{H} = \{h_0 \in V_\beta, h_0 \geq \chi_{k^*}\}$, then for all $(h_0, z) \in \mathfrak{H} \times \mathfrak{J}$, there holds $\tau(h_0, z) \leq t_*$. This finishes the proof of Lemma 2. \blacksquare

Closing remark 1. Ineluctability of blow-up for 1-corotational solutions of (SHMF') remains an open problem. Let $(\mathbb{P}_x)_{x \in \mathfrak{X}}$ be the Markovian family on $(\bar{W}, \mathcal{B}(\bar{W}))$ defined in the proof of Theorem 1. The following observation is made in [40, sec. 5] (note that we could also let \mathfrak{X} be any Polish space with additional isolated point $\{\Delta\}$).

For $w \in \bar{W}$ denote by $\tau(w) := \inf\{t \geq 0; w(t) = \Delta\}$. Assume that there exist $T > 0$, and an open set $B_0 \subset \mathfrak{X}$, such that:

Condition 1 (Uniform lower bound). There exists a constant p_0 , independent of $x \in B_0$ with $\mathbb{P}_x(\tau \leq T) \geq p_0$;

Condition 2 (Conditional recurrence). For all $x \in \mathfrak{X}$, $\mathbb{P}_x(\sigma = \infty \text{ and } \tau = \infty) = 0$, where for $w \in \bar{W}$, $\sigma(w)$ is defined as $\inf\{t \geq 0, w(t) \in B_0\}$.

Then for each $x \in \mathfrak{X}$

$$\mathbb{P}_x(\tau < \infty) = 1.$$

Taking $B_0 := \overset{\circ}{\mathfrak{H}}$, where \mathfrak{H} is as in Lemma 2, then Condition 1 has already been checked in (A.16): it suffices to let $p_0 := \mathbb{P} \circ Z|_{[0, T]}^{-1}(\mathfrak{J})$, where \mathfrak{J} is as in Lemma 2.

However Condition 2, i.e. the conditional recurrence for the pre-blow-up set $\overset{\circ}{\mathfrak{H}}$, seems difficult to check, because it relates large time behaviour of solutions of (SHMF'). A natural idea would be to replace first \mathfrak{H} by some neighbourhood \mathcal{V} of 0 in $C^1([0, 1])$, say, and then to bound below the probability to reach \mathfrak{H} from \mathcal{V} . In the deterministic case, such stability results are for instance obtained in [30] or [14] for the full LLG equation, and rely on the energy estimate $E(t) - E(0) + \iint_{[0, t] \times \mathbb{D}^2} |u \times \Delta u|^2 = 0$, which gives uniform bounds in $t > 0$. The main difficulty here is that the counterpart of the above identity writes:

$$E(t) - E_0 + \iint_{[0, t] \times \mathbb{D}^2} |u \times \Delta u|^2 dt = C_\phi t + M_\phi(t) \quad (3.30)$$

M_ϕ being a martingale, and C_ϕ a positive constant, but note that (3.30) is not sufficient to obtain uniform boundedness of $\frac{1}{t} \mathbb{E} \int_0^t E(s) ds$.

Closing remark 2. As already mentioned in Remark 1.1, it is not expected that the pre-blow-up sets remain open if we release the 1-corotational symmetry assumption. Consider maps with *two degrees of freedom*: $u_{g,h}(t, x) := (\cos g \sin h, \sin g \sin h, \cos h)$ where $x = (r \cos \theta, r \sin \theta)$, $g = g(t, r, \theta)$ and $h = h(t, r, \theta)$. Putting $u_{g,h}$ in (1.7) (which is statistically equivalent to (SHMF), at least formally), then we obtain the following parabolic system:

$$\begin{cases} dg = \left(\partial_{rr} g + \frac{\partial_r g}{r} + \frac{\partial_{\theta\theta} g}{r^2} + \frac{2}{\tan h} \left(\partial_r g \partial_r h + \frac{\partial_\theta g \partial_\theta h}{r^2} \right) \right) dt + \frac{1}{\sin h} \circ dw_1 \\ dh = \left(\partial_{rr} h + \frac{\partial_r h}{r} + \frac{\partial_{\theta\theta} h}{r^2} - \left(\partial_r g^2 + \frac{\partial_\theta g^2}{r^2} \right) \sin 2h/2 \right) dt + dw_2 \end{cases} \quad (3.31)$$

where $w_1(t, r, \theta), w_2(t, r, \theta)$ are independent.

The above conjecture gives some indication that blow-up phenomenon should not happen for (3.31), even if u is 1-equivariant, that is $g(t, r, \theta) = \theta + \tilde{g}(t, r)$ and where, in order to preserve this symmetry, we would take $w_j = w_j(t, r)$ for $j = 1, 2$. Non-constant $\tilde{g}(t, r)$ are shown in [32] to stabilize the solutions of the Heisenberg equation, which is related to the fact that the gyromagnetic term $u \times \Delta u$ makes the solution turn around the vertical axis $\vec{k} \equiv (0, 0, 1)$. This necessary extra degree of freedom also appears when taking the full noise term $dw_1 \circ \frac{\Theta_u}{\sin h} + dw_2 \circ \Phi_u$ in the equation. For this reason, we believe that finite-time blow-up for general solutions of (SHMF) is a zero-probability event.

A Appendix

A.1 Proof of the properties (2.7)-(2.8)-(2.9)

The eigenvectors of $(A, D(A))$ – see (1.11)-(1.12) – derive from the so called *Bessel functions of the first kind*. Recall that the order one Bessel function of the first kind, which is generally denoted by $J_1(y)$, $y \in \mathbb{R}^+$, is determined by the ODE:

$$\begin{cases} y^2 \frac{d^2 J_1}{dy^2} + y \frac{dJ_1}{dy} + (y^2 - 1)J_1 = 0, & \text{for } y \geq 0, \\ J_1(0) = 0. \end{cases}$$

The zeros of J_1 form a countable subset $(x_k)_{k \geq 1}$ of \mathbb{R}_+^* , and it is a well known fact that, if we arrange them in ascending order (we will do this assumption in the sequel), then the x_k 's are asymptotically linear in $k \in \mathbb{N}^*$. For $k \in \mathbb{N}^*$, the mappings

$$e_k := \left(r \mapsto \frac{1}{|J_1(x_k \cdot)|_H} J_1(x_k r) \right), \quad r \in I, \quad (A.1)$$

define a family $(e_k)_{k \in \mathbb{N}^*}$ of eigenvectors of A , with associated eigenvalues $-(x_k)^2$, $k \in \mathbb{N}^*$, which forms an orthonormal basis of H . It follows that A generates an analytical

semigroup of negative type $t \mapsto S(t) = e^{tA}$, $t \geq 0$, on the hilbert space H (see e.g. [37]). This provides also a precise definition of the fractional powers of $-A$, through

$$(-A)^{\beta/2} f := \sum_{k \in \mathbb{N}} (x_k)^\beta \langle f, e_k \rangle e_k, \text{ for } f \text{ in } V_\beta, \quad (\text{A.2})$$

the series being convergent in H , see (1.14).

Now, letting $\gamma := 2\alpha$, note that property (2.7) is a consequence of $|(-A)^{\gamma/2} h|_\beta^2 = \sum_k (x_k)^{2(\beta+\gamma)} e^{-2t(x_k)^2} \langle h, e_k \rangle^2$ together with the inequality

$$\sup_{x \geq 0} x^{2\gamma} e^{-2tx} \leq \left(\frac{\gamma}{t}\right)^{2\gamma} e^{-2\gamma}.$$

In order to prove (2.8)-(2.9), we need estimates on the nonlinear term $b(r, h) = (2h - \sin 2h)/(2r^2)$. The following interpolation Lemma is based on expansion of elements of H in terms of the so-called *Fourier-Bessel series* – see [44, chap. 18].

Lemma A.1. *Let $p \in [1, \infty]$, $\nu \in \mathbb{R}$.*

- (i) *Take $\nu \leq 2/p + 1$ and define the operator $T : D(T) \subset V_\beta \rightarrow L_{r \, dr}^p$ by $Tf = \varphi/r^\nu$ for $f \in D(T) := \{\varphi \in V_\beta, |\varphi/r^\nu|_{L_{r \, dr}^p} < \infty\}$.*

Provided $\beta > (1 + \nu - 2/p) \vee 1/2$, then T has a bounded extension

$$\begin{cases} T : V_\beta \longrightarrow L_{r \, dr}^p & \text{if } p < \infty \text{ and } \nu < 2/p + 1; \\ T : V_\beta \longrightarrow L^\infty & \text{if } p = \infty \text{ and } \nu \leq 1. \end{cases}$$

- (ii) *Similarly, the linear map $\partial_r : D(\partial_r) := \{\varphi \in V_\beta, |\partial_r \varphi|_{L_{r \, dr}^p} < \infty\} \longrightarrow L_{r \, dr}^p$, has a bounded extension $\partial_r : V_\beta \rightarrow L_{r \, dr}^p$, provided $\beta > (2 - 2/p) \vee 3/2$.*

Proof of Lemma A.1. According to (1.18) the following bound holds, provided $\beta > 2$:

$$\sup_{r \in I} \left\{ (\partial_r f(r))^2 + \frac{f(r)^2}{r^2} \right\} \leq c_\epsilon |f|_\beta^2, \quad \text{for all } f \in V_\beta,$$

for some $c_\beta > 0$. This yields both (i) and (ii) in the case $p = \infty$.

Proof of (i). Let $p \in [1, \infty)$. Using the orthonormal basis defined in (A.1), for $k \geq 1$, and setting $c_k := |J_1(x_k \cdot)|_H^{-1}$, one has by (A.1):

$$\left| \frac{1}{r^\nu} e_k \right|_{L_{r \, dr}^p}^2 = (c_k)^2 \left| \frac{1}{r^\nu} J_1(x_k \cdot) \right|_{L_{r \, dr}^p}^2 = (c_k)^2 (x_k)^{2\nu-4/p} \left(\int_0^{x_k} \frac{|J_1(y)|^p}{y^{p\nu}} y \, dy \right)^{2/p},$$

where we have done the change of variable $y = x_k r$. Using classical properties of Bessel functions, see [44, chap. 7], there exist constants $c, c' > 0$ such that

$$J_1(y) \leq cy, \quad \forall y \in I \quad \text{and} \quad |J_1(y)| \leq c' y^{-\frac{1}{2}}, \quad y \in [1, \infty). \quad (\text{A.3})$$

By (A.3), we formally obtain that for $\nu \in \mathbb{R}$,

$$\int_0^{x_k} \frac{|J_1(y)|^p}{y^{p\nu}} y \, dy \leq c'' \left(\int_0^1 y^{p-p\nu+1} \, dy + \int_1^{x_k} y^{-p/2-p\nu+1} \, dy \right). \quad (\text{A.4})$$

Since x_k is asymptotically linear in $k \geq 1$ ([44, p. 503-510]), the right hand side of (A.4) remains bounded independently of k if and only if $2/p - 1/2 < \nu < 2/p + 1$. Noticing furthermore that $(c_k)^2 \equiv |J_1(x_k \cdot)|_H^{-2} = \mathcal{O}(k)$ (this is left to the reader), we have

$$\left| \frac{1}{r^\nu} e_k \right|_{L_{rdr}^p}^2 = \begin{cases} \mathcal{O}(k^{1+2\nu-4/p}) & \text{if } \frac{2}{p} - \frac{1}{2} < \nu < \frac{2}{p} + 1, \\ \mathcal{O}(1) & \text{if } \nu \leq \frac{2}{p} - \frac{1}{2}. \end{cases}$$

Using now triangle and Cauchy-Schwarz inequalities on the Fourier-Bessel series of $f \in V_\beta$, we have formally

$$\|Tf\|_{L_{rdr}^p} \leq \sum_{k \geq 1} |\langle f, e_k \rangle| \|Te_k\|_{L_{rdr}^p} \leq \|f\|_\beta \left(\sum_{k \geq 1} (x_k)^{-2\beta} \|Te_k\|_{L_{rdr}^p}^2 \right)^{1/2}. \quad (\text{A.5})$$

Taking $\beta > 1 + \nu - 2/p$ gives a convergent series in (A.5) in the case $2/p - 1/2 < \nu < 2/p + 1$, whereas $\beta > 1/2$ is sufficient when $\nu \leq 2/p - 1/2$. In both cases, we obtain a continuous extension $T : V_\beta \rightarrow L_{rdr}^p$.

Proof of (ii). The second assertion works in the same way, using that $|\partial_r e_k|_{L_{rdr}^p} = c_k x_k |J_1'(x_k \cdot)|_{L_{rdr}^p}$. The well-known identity $J_1'(y) = J_0(y) - \frac{J_1(y)}{y}$, $y \geq 0$ (see [44, p. 17-19]), shows in particular that $J_1'(x_k \cdot)$ defines an element of L_{rdr}^p near the origin. For some constant $c > 0$ we obtain

$$\|\partial_r e_k\|_{L_{rdr}^p}^2 \leq c k^{3-4/p} \left(\|J_1'\|_{L_{rdr}^p([0,1])}^p + \int_1^{x_k} \left| J_0(y) - \frac{J_1(y)}{y} \right|^p y dy \right)^{2/p}. \quad (\text{A.6})$$

Now, as for J_1 there exists $c' > 0$, such that: $J_0 \leq c' y^{-1/2}$, the other term J_1/y being smaller at infinity. Therefore, in case $p > 4$, the integral in (A.6) is bounded, so that inequality (A.5), with $T := \partial_r$ and $\beta > 2 - 2/p$, gives the result. Otherwise if $p \in [1, 4]$, we have $\|\partial_r e_k\|_{L_{rdr}^p}^2 = \mathcal{O}(k^{4/p-1})$, and it is sufficient to take $\beta > 3/2$. ■

We can now turn to the proof of the remaining properties.

Proof of (2.8)-(2.8). Denoting by $F : x \mapsto x - \sin 2x/2$, $x \in \mathbb{R}$, and using the inequality $|F(x)| \leq c|x|^3$, $x \in \mathbb{R}$, for a certain $c > 0$, we have by an application of Lemma A.1-(i) with $\nu = 2/3$, $p = 6$:

$$|b(\cdot, v)|_H = \left| \frac{F(v)}{r^2} \right|_H \leq c \left| \frac{v}{r^{2/3}} \right|_{L_{rdr}^6}^3 \leq c' |v|_\beta^3,$$

as soon as $\beta > 4/3$, which shows (2.8).

Similarly, using that for some $c > 0$, $|F(x) - F(y)| \leq c|x - y|(x^2 + y^2)$, $\forall x, y \in \mathbb{R}$, then Hölder's inequality implies:

$$\begin{aligned} |b(\cdot, u) - b(\cdot, v)|_H &\leq c \left| \frac{u - v}{r^{2/3}} \right| \left| \left(\frac{u}{r^{2/3}} \right)^2 + \left(\frac{v}{r^{2/3}} \right)^2 \right|_H \\ &\leq c \left| \frac{u - v}{r^{2/3}} \right|_{L_{rdr}^6} \left(\left| \frac{u}{r^{2/3}} \right|_{L_{rdr}^6}^2 + \left| \frac{v}{r^{2/3}} \right|_{L_{rdr}^6}^2 \right). \end{aligned}$$

An application of Lemma A.1-(i) with the same parameters as above leads to (2.9). ■

A.2 Complements in the proof of Proposition 1: higher regularity

Local solvability when $\beta \in (2, 4]$. Take $2 < \beta \leq 4$, $h_0 \in V_\beta$ and assume that $\phi \in \mathbb{L}_2(H; V_\beta)$. By the same argument as above, we can fix $z \in C([0, T_*]; V_\beta)$ with $z(0) = 0$, and argue pathwise. Denote by $(h, \tau^{\beta-2})$, the maximal solution obtained in Section 2, which therefore belongs to $C([0, \tau^{\beta-2}); V_\beta)$. We aim to find an *a priori* bound on $\|h\|_{T, \beta}$ guaranteeing existence during a positive time. Write for $0 \leq t < \tau^{\beta-2}$:

$$h(t) = S(t)h_0 + \int_0^t (-A)^\delta S(t-s) \left[(-A)^{-1-\delta} (-Ab(\cdot, h)) \right] ds + z(t). \quad (\text{A.7})$$

where $\delta := (\beta - 2)/2 \in (0, 1)$, and using (2.7), we obtain the bound $|h(t)|_\beta \leq |S(t)h_0|_\beta + \int_0^t (t-s)^{-\delta} |Ab(\cdot, h)|_H ds + |z(t)|_\beta$, provided all terms are finite. Therefore, there remains to evaluate the term $|Ab(\cdot, v)|_H$. Direct computations lead to

$$\begin{aligned} Ab(r, h) = & \frac{1 - \cos 2h}{r^2} \partial_{rr} h \\ & + \frac{1 - \cos 2h}{r^3} \partial_r h - \frac{6h - 3 \sin 2h}{2r^4} \\ & + \frac{2 \sin 2h}{r^2} (\partial_r h)^2 + \frac{6h - 3 \sin 2h}{r^4} - \frac{4(1 - \cos 2h)}{r^3} \partial_r h, \end{aligned}$$

where, due to compensations, each line of the right hand side must be treated separately. Using the triangle inequality, we write for $h \in V_\beta$, $|Ab(r, h)|_H \leq I + II + III$, and deal with each term. For the sake of clarity, from now until the end of the proof, we use the notation $T_1(h) \lesssim T_2(h)$ if two terms involving $h \in V_\beta$ are comparable up to a multiplicative constant that does not depend on h .

In the sequel, we fix an arbitrary $\epsilon > 0$. Using the bound $|G(x)| \leq c|x|^2$, $x \in \mathbb{R}$, where $G : x \in \mathbb{R} \mapsto 1 - \cos(2x)$, Remark 1.2, and Lemma A.1–(i) in the case $v = 1$, $p = \infty$, the first term satisfies $I \lesssim |\frac{h}{r}|_{L^\infty}^2 |\partial_{rr} h|_H \lesssim |h|_{2+\epsilon}^2 |h|_2$, whereas for the second term we have:

$$\begin{aligned} II & \lesssim \left| \frac{h^2}{r^2} \left(\frac{\partial_r h}{r} - \frac{h}{r^2} \right) \right|_H + \left| \frac{1 - \cos 2h - 2h^2}{r^3} \partial_r h - \frac{3}{2} \left(\frac{2h - \sin 2h - (4/3)h^3}{r^4} \right) \right|_H \\ & = II_1 + II_2. \end{aligned}$$

Using Lemma A.1–(i) with $v = 1$, $p = \infty$, and Remark 1.2, there holds $II_1 \lesssim \left| \frac{h}{r} \right|_{L^\infty}^2 \left| \frac{\partial_r h}{r} - \frac{h}{r^2} \right|_H \lesssim |h|_{2+\epsilon}^2 |h|_2$. Moreover, by the classical inequalities $|1 - \cos 2x - 2x^2| \leq cx^4$, $|2x - \sin 2x - (4/3)x^3| \leq c|x|^5$ for $x \in \mathbb{R}$, Hölder inequality, and Lemma A.1–(i) with $(v, p) = (3/4, 40/3)$, and then (ii) with $p = 5$, (resp. (i) with $(v, p) = (4/5, 10)$), the following bound is obtained:

$$II_2 \lesssim \left| \frac{h^4}{r^3} \partial_r h \right|_H + \left| \frac{h^5}{r^4} \right|_H \lesssim \left| \frac{h}{r^{3/4}} \right|_{L_{rdr}^{40/3}}^4 |\partial_r h|_{L_{rdr}^5} + \left| \frac{h}{r^{4/5}} \right|_{L_{rdr}^{10}}^5 \lesssim |h|_{8/5+\epsilon}^5.$$

The bound on III is obtained in a similar way. We write that $III \leq III_1 + III_2$, with

$$\begin{aligned} III_2 & = \left| \frac{2 \sin 2h - 4h}{r^2} (\partial_r h)^2 + 3 \left(\frac{2h - \sin 2h - (4/3)h^3}{r^4} \right) - 4 \left(\frac{1 - \cos 2h - 2h^2}{r^3} \right) \partial_r h \right|_H \\ & \lesssim \left| \frac{h}{r^{2/3}} \right|_{L_{rdr}^{30}}^3 |\partial_r h|_{L_{rdr}^5}^2 + \left| \frac{h}{r^{4/5}} \right|_{L_{rdr}^{10}}^5 + \left| \frac{h}{r^{3/4}} \right|_{L_{rdr}^{40/3}}^4 |\partial_r h|_{L_{rdr}^5} \end{aligned}$$

which is bounded by $c|h|_{8/5+\epsilon}^5$, by the Sobolev embeddings of Remark 1.3. The main term III_1 has to be handled cautiously, since it involves typical compensations related to $D(A)$ (see (1.11)). Write that $III_1 = \left| \frac{h}{r^2} (\partial_r h)^2 + \frac{h^3}{r^4} - 2 \frac{h^2}{r^3} \partial_r h \right|_H = \left| \frac{h}{r} \partial_r h \left(\frac{\partial_r h}{r} - \frac{h}{r^2} \right) + \frac{h^2}{r^2} \left(\frac{h}{r^2} - \frac{\partial_r h}{r} \right) \right|_H$, so that

$$III_1 \lesssim \left| \frac{h}{r} \right|_{L^\infty} \left| \frac{\partial_r h}{r} - \frac{h}{r^2} \right|_H \left(\left| \partial_r h \right|_{L^\infty} + \left| \frac{h}{r} \right|_{L^\infty} \right) \lesssim |h|_2 |h|_{2+\epsilon}^2,$$

by Remark 1.2, and Lemma A.1–(i) with $(\nu, p) = (1, \infty)$.

Going back to (A.7), and fixing $\epsilon > 0$, we see that for some constant $c_\epsilon > 0$:

$$|h(t)|_\beta \leq c|h_0|_\beta + c_\epsilon \int_0^t (t-s)^{-\delta} g(s) ds + \|z\|_{\tau^{\beta-2}, \beta}, \quad t \in [0, \tau^{\beta-2}), \quad (\text{A.8})$$

where we let

$$g(s) := |h(s)|_{8/5+\epsilon}^5 + |h(s)|_{2+\epsilon}^2 |h(s)|_2. \quad (\text{A.9})$$

By a classical generalization of Grönwall Lemma, (A.8) implies existence in V_β for some positive time $0 < \tau^\beta \leq \tau^{\beta-2}$.

Propagation of regularity. Since the integrand $g(s)$ defined in (A.9) does not depend on $|h(s)|_\beta$, we see that $\tau^\beta \geq \tau^{2+\epsilon}$, and therefore:

$$\tau^{2+\epsilon} = \tau^\beta \text{ for every } 2+\epsilon \leq \beta < 4, \quad (\text{A.10})$$

and every $\epsilon > 0$. This ends the proof of Proposition 1. ■

A.3 Proof of Lemma 1: Continuous dependence of the solution $h(h_0, z)$ with respect to its arguments.

The following proof is adapted from that of [18, 19]. In the sequel, we fix $\beta \in (4/3, 2]$, $h_0 \in V_\beta$ and $z \in C([0, T]; V_\beta)$. For $R, T > 0$, we denote by \mathbb{B}_T^R (resp. B^R) the ball of radius R , centered at z in $C([0, T]; V_\beta)$ (resp. h_0 in V_β). If $(h_1, \zeta) \in V_\beta \times C([0, T]; V_\beta)$, we will denote by $v(h_1, \zeta, \cdot)$ the corresponding (maximal) mild solution of (1.10), obtained by reiteration of the fixed point argument for $\Gamma_{h_1, \zeta, T}$ (see Section 2). We also denote by $\tau(h_1, \zeta)$ its existence time.

Proof of Lemma 1. Assume that $\tau(h_0, z) > T$, and let

$$R := \|v(h_0, z)\|_{T, \beta} \vee \|z\|_{T, \beta} + 1, \quad (\text{A.11})$$

define $T_*(R)$ as in (2.12), and set $N := \lfloor T/T_* \rfloor$. We prove the result by induction. For each $k \in \{1, \dots, N\}$ denote by (H_k) the sentence:

(H_k) . “There exists $\delta_k > 0$, such that if $(h_0, z) \in B^{\delta_k} \times \mathbb{B}_T^{\delta_k}$, then: $\tau(h_0, z) > kT_*$, and the map $(h_0, z) \in B^{\delta_k} \times \mathbb{B}_{kT_*}^{\delta_k} \mapsto v(h_0, z, kT_*)$ is continuous.”

The case $k = 1$ has been proved in Section 2: it suffices to take $\delta_1 > 0$ depending on R only, so that (2.13) holds for all $(h_1, \zeta) \in B^R \times \mathbb{B}_{T_*}^R$.

Inductive step. Let $k \geq 1$ and assume $(H_\ell)_{0 \leq \ell \leq k}$. In particular (H_k) implies the existence of $\delta > 0$, such that $|\nu(h_1, \zeta, kT_*) - \nu(h_0, z, kT_*)|_\beta < \delta_1$ for every $(h_1, \zeta) \in B^\delta \times \mathbb{B}_{kT_*}^\delta$. For $t \in [0, T_*]$, denoting by $x(t) := z(t + kT_*) - S(t)z(kT_*)$, by $\xi(t) := \zeta(t + kT_*) - S(t)\zeta(kT_*)$, and assuming without loss of generality that $\delta < \delta_1/2$, we have $\|\xi - x\|_{T_*, \beta} < \delta_1$. By (H_1) , this implies that $\nu(h_1, \zeta, \cdot)$ is at least defined up to $(k+1)T^*$. Moreover, by uniqueness:

$$\nu(h_1, \zeta, (k+1)T_*) = \nu(\nu(h_1, \zeta, kT_*), \xi, T_*). \quad (\text{A.12})$$

Still by (H_1) , (A.12) defines a continuous map with respect to $(h_1, \zeta) \in B^{\delta_k \wedge \delta} \times \mathbb{B}_{(k+1)T_*}^{\delta_k \wedge \delta}$. This proves (H_{k+1}) , letting $\delta_{k+1} := \delta_k \wedge \delta$.

In particular, (H_N) is true, which implies the proposition when $\beta \in (4/3, 2]$. Higher regularity is standard. \blacksquare

A.4 Proof of the comparison principle

For $J := [0, r_1] \subset I$, we denote the parabolic boundary by $\Sigma_\kappa := \{0\} \times J \cup [0, \kappa) \times \partial J$. To avoid cumbersome computations, when $f \in H$ we denote by $\int_J f := \int_J f(r) r dr$, and fixing z as in (3.2) we write

$$q_f(t, r) := f(t, r) + p(z(t, r) + f(t, r)), \quad (t, r) \in [0, \kappa) \times J.$$

Take now $0 < T < \kappa$, and let $t \mapsto \zeta(t, \cdot) \in C([0, T] \times [0, r_1])$ be a non-negative map such that $\zeta(t, r)$ vanishes for $(t, r) \in \Sigma_T$. Using (i) and (ii), we obtain

$$-\int_0^t \int_J (f - g) \partial_t \zeta \leq -\int_0^t \int_J \partial_r (f - g) \partial_r \zeta + \frac{(q_f - q_g) \zeta}{r^2}. \quad (\text{A.13})$$

Note that if $\varphi \in V_2$, and $\psi \in V_1$, there holds the integration by parts formula: $\langle -A\varphi, \psi \rangle = \langle \partial_r \varphi, \partial_r \psi \rangle + \langle \frac{\varphi}{r}, \frac{\psi}{r} \rangle$, so that $V_1 = D((-A)^{1/2})$ corresponds to the space $\{h \in H, \frac{h}{r} \in H, \partial_r h \in H\}$. Due to (3.2)-(3.3), and because of $f, g \in C([0, T]; V_1)$, then the right hand side of (A.13) is bounded by $c\|f - g\|_{T,1} \|\zeta\|_{T,1}$. By density (A.13) can thus be extended to the larger class of test functions

$$\mathcal{T} := \{\zeta : [0, T] \times J \rightarrow \mathbb{R}_+, \zeta|_{\Sigma_T} = 0 \text{ and } \|\zeta\|_{T,1} + \|\partial_t \zeta\|_{T,0} < \infty\}.$$

Denote by $[x]_+ := \max\{x, 0\}$, and define $\zeta(t, r) := [f - g]_+(t, r)$. The fact that $f, g \in C^1([0, T]; H)$ implies

$$\frac{d}{dt} \int_J [f - g]_+^2 = 2 \int_J \partial_t (f - g) \zeta, \quad (\text{A.14})$$

which is summable on $[0, T]$. Noticing furthermore that $\zeta \in \mathcal{T}$ (note that $f \in V_1 \Rightarrow [f]_+ \in V_1$), applying (A.13) to ζ , (A.14) and integrating by parts gives:

$$\begin{aligned} \frac{1}{2} \int_J [f(t) - g(t)]_+^2 &\leq - \iint_{[0,t] \times J} \mathbb{1}_{f \geq g} (\partial_r (f - g))^2 + \frac{[f - g]_+ (q_f - q_g)}{r^2} \\ &\leq - \iint_{[0,t] \times J} \frac{[f - g]_+ (q_f - q_g)}{r^2}, \end{aligned} \quad (\text{A.15})$$

where we have used the fact that the weak derivative of $x \in \mathbb{R} \mapsto [x]_+$, is the map $x \in \mathbb{R} \mapsto \mathbb{1}_{\mathbb{R}_+}(x)$. By (3.2)-(3.3), and since $\beta > 1$ (implying the uniform continuity of z, f, g

on compacts, see Remark 1.3), we can find $\varepsilon(t, r)$ depending on $p''(0), f, g$, such that $\varepsilon(t, r) \rightarrow 0$ as $r \rightarrow 0$, uniformly in $t \in [0, T]$, and such that $q_f(t, r) - q_g(t, r) = (1 + p'(0) + \varepsilon(t, r))(f(t, r) - g(t, r))$. Since $p'(0) > -1$ and $f|_\Sigma \leq g|_\Sigma$, this yields the existence of $\bar{r} = \bar{r}(T)$ such that:

$$[f - g]_+(q_f - q_g) \geq 0 \text{ for a.e. } (t, r) \in [0, T] \times [0, \bar{r}]. \quad (\text{A.16})$$

Finally we write for all $t \in [0, T]$:

$$\begin{aligned} \int_J [f(t) - g(t)]_+^2 \leq & - \iint_{[0, t] \times [0, \bar{r}]} \frac{[f - g]_+(q_f - q_g)}{r^2} \\ & + \frac{1}{2\bar{r}^2} \iint_{[0, t] \times [\bar{r}, 1]} [f - g]_+ |q_f - q_g|, \end{aligned}$$

which by (A.16) and (3.3), is bounded by $\frac{K}{\bar{r}^2} \iint_{[0, t] \times J} [f - g]_+^2$. We finally obtain $\| [f - g]_+ \|_H^2(t) \leq \frac{K}{\bar{r}^2} \int_0^t \| [f - g]_+ \|_H^2(s) ds$ for $t \in [0, T]$, and $[f - g]_+|_{[0, T] \times J} \equiv 0$ follows by Gronwall Lemma. Reiterating on every subinterval $[0, T] \subset [0, \kappa)$ gives $f \leq g$ on $[0, \kappa)$. ■

References

- [1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume **140** of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
- [2] F. Alouges, A. De Bouard, and A. Hocquet. A semi-discrete scheme for the stochastic Landau–Lifshitz equation. *Stochastic Partial Differential Equations: Analysis and Computations*, **2**(3):281–315, 2014.
- [3] F. Alouges and A. Soyeur. On global weak solutions for Landau–Lifshitz equations: existence and nonuniqueness. *Nonlinear Analysis: Theory, Methods & Applications*, **18**(11):1071–1084, 1992.
- [4] L. Bañas, Z. Brzeźniak, M. Neklyudov, and A. Prohl. A convergent finite-element-based discretization of the stochastic Landau–Lifshitz–Gilbert equation. *IMA Journal of Numerical Analysis*, page drt020, 2013.
- [5] L. Bañas, Z. Brzeźniak, M. Neklyudov, and A. Prohl. *Stochastic Ferromagnetism—Analysis and Numerics*. De Gruyter, 2013.
- [6] L. Bañas, Z. Brzeźniak, and A. Prohl. Computational Studies for the Stochastic Landau–Lifshitz–Gilbert Equation. *SIAM Journal on Scientific Computing*, **35**(1):B62–B81, 2013.
- [7] J. B. v. d. Berg and J. Williams. (In-) Stability of Singular Equivariant Solutions to the Landau–Lifshitz–Gilbert Equation. *arXiv preprint arXiv:1107.2620*, 2011.
- [8] D. Berkov. Magnetization dynamics including thermal fluctuations. In H. Kronmüller and S. Parkin, editors, *Handbook of Magnetism and Advanced Magnetic Materials*, volume **2**, pages 795–823. Wiley Online Library, 2007.

- [9] M. Bertsch, R. Dal Passo, and R. van der Hout. Nonuniqueness for the Heat Flow of Harmonic Maps on the Disk. *Archive for rational mechanics and analysis*, **161**(2):93–112, 2002.
- [10] W. F. Brown. *Micromagnetics*. Interscience, New York, 1963.
- [11] W. F. Brown. Thermal fluctuations of a single-domain particle. *Physical Review*, **130**(5):1677, 1963.
- [12] Z. Brzeźniak, B. Goldys, and T. Jegaraj. Large deviations for a stochastic Landau–Lifshitz equation, extended version. *arXiv preprint arXiv:1202.0370*, 2012.
- [13] Z. Brzeźniak, B. Goldys, and T. Jegaraj. Weak solutions of a stochastic Landau–Lifshitz–Gilbert equation. *Applied Mathematics Research eXpress*, **2013**(1):1–33, 2013.
- [14] G. Carbou and P. Fabrie. Comportement asymptotique des solutions faibles des équations de Landau–Lifschitz. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, **325**(7):717–720, 1997.
- [15] K.-C. Chang, W. Y. Ding, R. Ye, et al. Finite-time blow-up of the heat flow of harmonic maps from surfaces. *Journal of Differential Geometry*, **36**(2):507–515, 1992.
- [16] J.-M. Coron and J.-M. Ghidaglia. Équations aux dérivées partielles. Explosion en temps fini pour le flot des applications harmoniques. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, **308**(12):339–344, 1989.
- [17] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*. Cambridge University Press, 2008.
- [18] A. De Bouard and A. Debussche. On the effect of a noise on the solutions of the focusing supercritical nonlinear Schrödinger equation. *Probability theory and related fields*, **123**(1):76–96, 2002.
- [19] A. De Bouard and A. Debussche. The stochastic nonlinear Schrödinger equation in H^1 . 2003.
- [20] A. De Bouard and A. Debussche. Blow-up for the stochastic nonlinear Schrödinger equation with multiplicative noise. *Annals of probability*, **33**(3):1078–1110, 2005.
- [21] A. Debussche, M. Hofmanová, and J. Vovelle. Degenerate parabolic stochastic partial differential equations: Quasilinear case. *arXiv preprint arXiv:1309.5817*, 2013.
- [22] M. Dozzi and J. A. López-Mimbela. Finite-time blowup and existence of global positive solutions of a semi-linear SPDE. *Stochastic Processes and their Applications*, **120**(6):767–776, 2010.
- [23] J. Eells and L. Lemaire. A report on harmonic maps. *Bulletin of the London Mathematical Society*, **10**(1):1–68, 1978.

- [24] J. Eells and L. Lemaire. Another report on harmonic maps. *Bulletin of the London Mathematical Society*, **20**(5):385–524, 1988.
- [25] J. Eells and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *American Journal of Mathematics*, pages 109–160, 1964.
- [26] T. L. Gilbert. A Lagrangian formulation of the gyromagnetic equation of the magnetization field. *Phys. Rev.*, **100**:1243, 1955.
- [27] B. Goldys, L. K.N., and T. T. A finite element approximation for the stochastic Landau–Lifshitz–Gilbert equation. 2013.
- [28] M. Hairer, M. D. Ryser, and H. Weber. Triviality of the 2D stochastic Allen-Cahn equation. *Electron. J. Probab*, **17**(39):1–14, 2012.
- [29] R. S. Hamilton. *Harmonic maps of manifolds with boundary*, volume **471**. Springer, 1975.
- [30] C. Kung-Ching. Heat flow and boundary value problem for harmonic maps. In *Annales de l’IHP Analyse non linéaire*, volume **6**, pages 363–395, 1989.
- [31] L. D. Landau and E. M. Lifshitz. On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. *Phys. Z. Sowjetunion*, **8**(153):101–114, 1935.
- [32] F. Merle, P. Raphaël, and I. Rodnianski. Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map. *Comptes Rendus Mathématiques*, **349**(5):279–283, 2011.
- [33] C. Mueller. The critical parameter for the heat equation with a noise term to blow up in finite time. *Annals of probability*, pages 1735–1746, 2000.
- [34] C. Mueller and R. Sowers. Blowup for the heat equation with a noise term. *Probability theory and related fields*, **97**(3):287–320, 1993.
- [35] L. Néel. Bases d’une nouvelle théorie générale du champ coercitif. In *Annales de l’Université de Grenoble*, volume **22**, pages 299–343, 1946.
- [36] M. Neklyudov and A. Prohl. The role of noise in finite ensembles of nanomagnetic particles. *Archive for Rational Mechanics and Analysis*, **210**(2):499–534, 2013.
- [37] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York, 1983.
- [38] P. Raphael and R. Seiwinger. Stable Blowup Dynamics for the 1-Corotational Energy Critical Harmonic Heat Flow. *Communications on Pure and Applied Mathematics*, **66**(3):414–480, 2013.
- [39] M. G. Reznikoff. *Rare events in finite and infinite dimensions*. PhD thesis, New York University, 2004.
- [40] M. Romito. Uniqueness and blow-up for the noisy viscous dyadic model. *arXiv preprint arXiv:1111.0536*, 2011.

- [41] M. Struwe. On the evolution of harmonic mappings of Riemannian surfaces. *Commentarii Mathematici Helvetici*, **60**(1):558–581, 1985.
- [42] J. B. Van Den Berg, J. R. King, and J. Hulshof. Formal asymptotics of bubbling in the harmonic map heat flow. *SIAM Journal on Applied Mathematics*, **63**(5):1682–1717, 2003.
- [43] J. B. Van den Berg and J. Williams. (In-)stability of singular equivariant solutions to the Landau–Lifshitz–Gilbert equation. *European Journal of Applied Mathematics*, **24**(06):921–948, 2013.
- [44] G. N. Watson. *A treatise on the theory of Bessel functions*. Cambridge university press, 1995.